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THE AMERICAN MATHEMATICAL MONTHLY

FOUNDED BY BENJAMIN F. FINKEL

A JOURNAL FOR TEACHERS OF MATHEMATICS IN THE COLLEGIATE AND ADVANCED SECONDARY FIELDS

PUBLISHED WITH THE COOPERATION OF

THE UNIVERSITIES OF CHICAGO, ILLINOIS, INDIANA, IOWA, KANSAS, MICHIGAN, MINNESOTA, MISSOURI, NEBRASKA; COLORADO COLLEGE AND OBERLIN COLLEGE

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THE

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ANNOUNCEMENT

With this issue the AMERICAN MATHEMATICAL MONTHLY passes into the control of an Editorial Board consisting of representatives of eleven supporting institutions, together with B. F. FINKEL, founder of the MONTHLY and editor since its inception in 1894.

It will be the editorial policy of the MONTHLY to appeal especially to teachers of mathematics in the collegiate and advanced secondary fields, not only for the purpose of directing attention to questions of improvement in teaching in these fields, but also to foster the development of the scientific spirit among large numbers who are not now reached by the more technical journals. The MONTHLY will continue to publish original articles, but increased attention will be given to pedagogical and historical questions of interest and value to teachers of collegiate mathematics.

The constituency of such a journal should include:

1) All teachers of the advanced courses in secondary schools, especially those which offer trigonometry, college algebra, and analytic geometry.

2) All teachers of undergraduate courses in mathematics in colleges, universities, and engineering schools.

3) University professors of mathematics who wish to keep in touch with pedagogical movements in the collegiate field.

4) Graduate students in mathematics who wish to profit by historical and pedagogical discussions among teachers of experience.

5) All productive workers in mathematics who may occasionally desire a place of publication for articles of minimum technical difficulty suitable for the promotion of scientific interest among the average mathematical readers.

6) All who are interested in the proposal and solution of problems, especially those who seek assistance from co-workers with respect to actual difficulties encountered in the prosecution of research.

The need of a standard journal with such aims, in a field otherwise unoccupied in this country, will not be questioned. But its ultimate success will depend upon its speedily becoming self-supporting. This can happen only if the individuals who should form its constituency become subscribers in large numbers. In order to make this possible the subscription price will remain only *$2.00 per year in advance.*

This copy of the MONTHLY is sent to all present subscribers, and also to a selected list of persons who, it is believed, will desire to become subscribers. The mailing list for the February issue will consist of those whose subscriptions, or renewals, are received by the Treasurer before the February number goes to press.

PLEASE FILL OUT THE ACCOMPANYING SUBSCRIPTION BLANK AND MAIL IT NOW

ORGANIZATION OF THE EDITORIAL BOARD

The Board of Editors of the MONTHLY is organized under Articles of Agreement in accordance with which the officers and the editorial and executive committees are elected by majority vote of all the members, the other committees being selected by the managing editor.

The names of the officers and committee members are given in this issue in the various departments, and they are here collected for convenient reference.

The members of these committees will welcome suggestions and criticisms from all sources, especially from those who are interested in helping the MONTHLY to realize its ideals.

The regular edition of the MONTHLY, as at present planned, is to contain thirty-two pages and cover. The present number is somewhat larger.

It is intended to go to press on the fifteenth of the month, as nearly as possible. The delay in this issue was unavoidable on account of the many adjustments incident to the reorganization.

The annual volume will consist of ten numbers, none being issued in the months of July and August.

THE AMERICAN MATHEMATICAL MONTHLY

VOLUME XX | JANUARY, 1913 | NUMBER 1

FOREWORD ON BEHALF OF THE EDITORS.

By E. R. HEDRICK, University of Missouri.

Collegiate Mathematics.—The lifework of hundreds of men is the teaching of collegiate mathematics; hundreds of other men are preparing for this same lifework. Thousands of students are now working and will work under the guidance of these men in collegiate courses in mathematics.

Suggestions of moment arise in the minds of many, yet there exist few means for transmitting an idea in this field to others. Any organized effort toward the dissemination of methods that are new, or of facts that are interesting, in this field is totally lacking.

These remarks apply, of course, only to the United States. Abroad, in every country in which mathematics holds a place commensurate with its position here, definite journals exist whose purpose is almost wholly to foster these interests. The AMERICAN MATHEMATICAL MONTHLY, beginning with this issue, proposes to afford an opportunity for any discussions that seem valuable upon collegiate mathematics, and the editors invite contributions concerning the methods of instruction as well as those that treat special topics or theorems.

Problems of Instruction.—Every teacher in this field has asked himself what topics he should best teach, why these topics are taught, how they should be approached, what purpose should be recognized as the main aim of collegiate mathematics. Ordinarily he has either found no answer whatever and has settled down to the basis of traditional procedure, or else he has satisfied himself perforce with partial answers that he could himself discover. Each man has been a power unto himself, but no man has received or given aid, at least to the extent of his ability.

Some conspicuous exceptions to this general state of affairs are mentioned below. What has been done is, however, sporadic, discontinuous, and of little influence,—certainly no such effort has been general, and the effects of what has been attempted are far from general.

The most general and most sustained work in this direction has recently culminated in the report of a committee of twenty mathematicians and engineers, under the chairmanship of Professor E. V. Huntington, of Harvard University.

This committee, was originally appointed at a joint meeting* of mathematicians and engineers, organized in 1907 by the Chicago Section of the American Mathematical Society; but it was authorized to report to the Society for the Promotion of Engineering Education, which body had already carried on a consistent series of discussions through a period of years on the teaching of mathematics to students of engineering; and these engineers, rather than the mathematicians, have furnished the medium, through their journal, for the publication of these discussions and of this final report.†

Though it may be contended that the aims of this Society are at least special, and touch only one side of our work, and though the spirit of many of the discussions may seem narrow to many, it is at least true that the attempt to suggest means for the improvement of mathematical teaching has been carried on consistently, courageously, and earnestly.

The efforts of this society, whose special aim is the improvement of engineering education, emphasize the variety of conditions that must be met by collegiate courses in mathematics at this time. The reasonable demands of the rapidly increasing schools of engineering, together with the normally increasing enrollments in academic classes, have vastly increased the problems that confront us.

Origin of New Problems.—The problems of to-day differ from those of a generation ago on account of three great movements that lie wholly outside the field of mathematics. Of these, the first is the spread and almost universal acceptance of the elective principle in colleges and in academic classes in institutions of every kind. To be sure, the tendency to-day seems to be to limit the scope of the elective system, but the days of the old fixed curriculum, with mathematics as a required study for all students, has passed, and forever.

Another movement of no less extent is the rapid increase in the number of those entering engineering and other semi-technical professions. In schools of engineering, a certain amount of mathematics is required of all students, though the amount varies. But this requirement that all students take collegiate courses in mathematics is not made, as was the older academic requirement, for the sake of general mental discipline and the training of the mind as a whole. While this motive may persist in part, the recent discussions by engineers, mentioned above, leave no room for doubt that the inherent reason, in this instance, is the belief in the direct usefulness to the engineer of the facts, the principles, and the processes to be learned in these courses.

The last, in point of time, of the three principal movements mentioned above is the present rapid change in educational theory, which is based to a large extent on experimental psychology, and which bears the same relation to the older pedagogy that chemistry bears to alchemy. Not all its problems are yet settled,

* A few printed copies of the papers read at this meeting may still be had from the secretary of the Chicago Section, H. E. Slaught, 58th Street and Ellis Ave., Chicago, Ill.

† This final report is in the form of a syllabus of courses in mathematics for students of engineering, and is for sale by the Society for the Promotion of Engineering Education, 43 E. Nineteenth St., New York.

many of them are now under intense discussion, but the manner of that discussion is the manner of science rather than the manner of controversy. One of its contributions that is widely accepted is that there are no such *fixed* faculties of the mind as those we had believed in rather implicitly; and that the training of such supposed faculties as the memory, for example, is impossible in its widest sense because there seems to be no one such faculty. The memory for number, for example, seems to be wholly unrelated to the memory for poetry, or the memory for forms.

Discipline.—To what extent training in any field carries over into any other field seems to be still under discussion. Whether the memory, for example, extends at all beyond the very region of the training received, whether a child who has been drilled in one special field gains in ability in another even closely allied field, seems yet not precisely known. That upon which there is complete agreement is that training of such a supposed faculty as the memory in *one* field does not strengthen the memory in *all* fields, but that it does strengthen the memory in at least some fields closely allied to that in which the training was received. The uncertainty that remains is to define how close the alliance must be.

While the question of training the memory is itself at least a minor consideration in the minds of many teachers of mathematics, the very analogous question of disciplinary training of the mind has been and is one of the strongest motives in the minds of many for the teaching of any course in mathematics. It was certainly the basis for the older requirement of mathematics in the early colleges. In so far as the modern theory of education renders uncertain the possibility of disciplinary training, it vitally affects every discussion of any mathematical topic. In how far training of the reasoning faculty may aid the student to better reasoning in general, in how far power to think in a particular field may be generally efficacious, are, in the opinion of many, problems whose solution lies in the future. We only know that the training received in reasoning, or thinking, upon one topic affects most strongly the ability to reason, or to think, on topics most closely associated with the topics in which the training was received. The uncertainty that is recognized is in describing accurately how close to the original topic a new one must be in order to show the effect of the previous training.

Selection of Topics.—Teachers of mathematics will not lightly abandon the value of the disciplinary training afforded by the subject, but the ability and scientific attitude of mind of teachers of collegiate mathematics, as compared with teachers in more elementary schools, will lead them to search for the truth about these questions and will make them cautious in basing their faith entirely upon an aim that is made questionable by these modern investigations in education.

One immediate result is the realization that the same amount of mental training, whatever that may be, seems likely to result from any one of several possible topics, if each of them is studied with the same thoroughness and atten-

tion. Does this make possible the selection of topics on grounds entirely different from mental discipline and, if the topics to be taught are selected for other reasons, will the resultant training be just as efficient?

This consideration, and the changed conditions which seem to raise the same demand, would point toward a very thorough and critical study of the motives, other than disciplinary training, for the study of mathematical courses, and of topics within those courses. Without discarding discipline as one of the desirable results of a mathematical training, a knowledge of other motives for the study of any one topic will serve only to enhance its value.

Comparative merits of different topics and of different methods of instruction can be realized most effectively if these other motives are considered, and if the disciplinary factors are temporarily ignored, for the claims of disciplinary value, as between two topics, are at best conflicting and uncertain, and the wholesome effect of any topic that is studied intensively will be recognized by all.

Interest.—The qualification that a study of any topic, in order to be effective, must be intensive and earnest will be insisted upon by all. It would seem therefore that profitable discussion may well take place concerning the means by which such work can be obtained. To arouse the student to his maximum effort, to stimulate his greatest efficiency, to gain the fullest measure of disciplinary good, there may be many plans of merit; but it is certain that all of them will have in common the holding of the student's attention, without which no effective study is thinkable. Any process for holding the attention in a sustained and serious fashion is described by saying that the student has become "interested" in the study or that his "interest" has been secured. But the word interest has been misused. The doctrine of interest has led to a mistaken doctrine of "amusement," than which no interpretation of the doctrine of interest could be more absurdly irrelevant. It is scarcely to be wondered at that the whole doctrine, sound as its real basis may be, has been misunderstood, abused, ridiculed, and that it appears to many as the antithesis of that very doctrine of discipline of which it is the sole effective support.

The Province of the Monthly.—This article is not itself a pedagogical document. What is here stated is perfectly well known in pedagogical circles, and we trust that we have not through ignorance, as we have certainly not through intention, overstepped the bounds of that which is generally recognized as finally established in the theory of education.

It is not the province of the MONTHLY to enter the field of general pedagogy, nor will the MONTHLY entertain discussions that are concerned with research in general pedagogy, or that deal with new theories of pedagogy, however important these contributions may seem. Journals for the publication of such contributions abound, and any article of worth in the general field is already more than liberally provided with possible means of publicity. In such matters, we should not venture to pose as expert critics of new matter, nor to pass upon the merits of articles. We must accept, as from a foreign field, the results of general pedagogy and of experimental psychology, and in order to be secure, we

must remain always a little behind, rather than in advance of the times, on pedagogical theory.

What we do desire is to inspire, not a discussion along these lines, but rather a discussion of definite *mathematical* problems. It is our hope that this article will spur mathematicians to the realization of the new conditions that confront us, to the variety of problems that demand discussion, to the vital importance which their solution holds to a great body of intelligent people. For the discussion from this standpoint of mathematical questions, properly speaking, the MONTHLY will open its columns without reserve, except for those usual discretionary and advisory powers that are traditional in any editorship.

The Future.—This is not the last word that the MONTHLY will have to say upon this question. We shall indulge in rather little general pedagogical preaching, for we claim no professional attitude toward that subject. What seemed necessary has been said above. For this issue this is enough. We shall, however, not place any limitation upon the scope of our suggestions to readers and to possible contributors; and we shall hope, by these suggestions, to point out very explicit problems of a mathematical nature which might well be treated. We shall aim to awaken an interest in their discussion, even among those who now regard their own profession—the teaching of collegiate mathematics—with distrust as a possible field for that type of human thinking that is known as scientific research. In this distrust, we, the editors of the MONTHLY, emphatically announce that we do not share.

HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

By FLORIAN CAJORI, Colorado College, Colorado Springs.

I. FROM NAPIER TO LEIBNIZ AND JOHN BERNOULLI I. 1614–1712.

LOGARITHMS OF POSITIVE NUMBERS.

The seventeenth and eighteenth centuries are notable in the history of algebra for important developments of the notations and ideas which mark its modern treatment. The study of this period in algebra is more than mere pastime. In the words of Felix Klein: "If we really desire to advance to a full understanding of the theory of logarithms, it is best to follow in broad outline the history of its creation."[1]

Logarithms were invented before our modern exponential notation, a^n, was introduced into algebra. To be sure, the use of algebraic symbols that were more or less different from the modern symbols to indicate powers and roots of a number had been suggested before the advent of the logarithm, but we shall

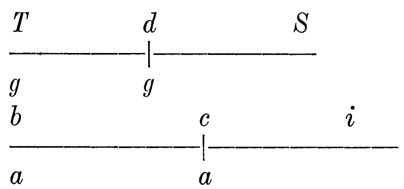
[1] F. Klein, *Elementarmathematik vom höh. Standpunkte aus*, I, Leipzig, 1908, p. 325.

see that these suggestions had remained unheeded; the fact is that the inventors of logarithms did not use the modern exponential notation and were not familiar with the exponential concept which now plays such a fundamental rôle in the development of logarithmic theory. What then were the basic considerations in the development of logarithms as entertained by their inventors, John Napier and Joost Bürgi?

John Napier's *Mirifici logarithmorum canonis descriptio*[1] appeared in 1614 in Edinburgh, and his *Mirifici logarithmorum canonis constructio* appeared there as a posthumous work in 1619, though written as early as, or earlier than, the *Descriptio.* Napier based his explanations upon two considerations: (1) The geometrico-mechanical concept of flowing points, (2) the relations which exist between arithmetic and geometric series. Several writers before the time of Napier called attention to certain relations between the terms of a geometric series and the terms of an arithmetic series, which relations involve the logarithmic idea. But those writers did not realize the possibilities of this idea nor did they conceive and execute the plan of computing a pair of corresponding series sufficiently dense for practical use in computation. From certain passages in authors like Stifel[2] one might be tempted to say that the logarithmic concept really existed before the time of Napier and Bürgi. Yet how much of a novelty the logarithms of Napier really were to the foremost mathematicians of his day can be realized by the enthusiasm with which men like Briggs and Kepler took up the new topic. So new and original did logarithms seem to Briggs that he left London for Scotland, to visit Napier, the inventor. Briggs addressed him thus:

> "My Lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy viz., the logarithms."

Napier lets the point g move along the definite line TS with a diminishing velocity such that its velocity at T is to that at d, as the distance TS is to the distance dS. At the same time Napier lets a point a move along the line bc (which is of indefinite length) with a uniform velocity which is the same as the initial velocity of the point g. If the two points start to move at the same moment, and if g is at d when a is at c, then the length bc is defined as the logarithm of dS. Napier constructed tables for trigonometric computation. With that end in view he lets TS stand for the *radius*, assigning to it the value 10^7, while dS stands for a given *sine*. At that time trigonometric functions were not thought of strictly as *ratios*. In the language of Napier, the definition of a logarithm is here as follows:



[1] A reprint of the entire *Descriptio* is found in Maseres' *Scriptores logarithmici*, Vol. 6, London, 1807, pp. 475–624.

[2] Tropfke, *Gesch. d. Elementar-Mathematik*, Vol. II, 1903, pp. 141–145.

"The logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which radius began to decrease geometrically, and in the same time as radius has decreased to the given sine."[1]

Letting $v = 10^7$, the geometric and arithmetic series of Napier may be exhibited in modern notation as follows:

$$\begin{array}{cccccc} v, & v(1-1/v), & v(1-1/v)^2, & \cdots & v(1-1/v)^v & \cdots \\ 0, & 1\quad, & 2\quad, & \cdots & v & \cdots \end{array}$$

The numbers in the upper series represent successive values of the *sines;* the numbers in the lower series stand for the corresponding logarithms. Thus, $\log 10^7 = 0$, $\log (10^7 - 1) = 1$, and generally, $\log (10^7\{1 - 10^{-7}\})^n = n$, where $n = 0, 1, 2, \cdots$. Napier had a definite object in view in making the logarithm of the radius, 10^7, equal to zero. He looked upon this arrangement as labor-saving, considering the frequency with which multiplications and divisions by the radius arise in trigonometric computation. He made the geometric series decrease while the corresponding arithmetic series increases. This makes the logarithms of the sines of all angles between 0° and 90° positive; in our modern tables these logarithms are negative.

In the *Descriptio* logarithms are defined as follows: *Logarithmi sunt numeri qui proportionalibus adjuncti aequales servant differentias.*" (Logarithms are numbers which correspond to proportional numbers and have equal differences.) The proportional numbers are the terms of the geometric progression; the numbers having equal differences are the terms of the arithmetic progression. The word "logarithm" is of Greek structure and signifies "number of ratios." The idea is this: $v(1 - 1/v)^n$ is gotten from v by n successive applications of the ratio $(1 - 1/v)$. Hence n, which is the logarithm, indicates "the number of ratios." Napier restates the definition in 1614 as follows: *Logarithmi dici possunt numerorum proportionalium comites æquidifferentes.* (Logarithms may be called equidifferent companions to proportional numbers.)

Not only is Napier's definition of a logarithm different from the modern definitions, but the notion of a "base" is inapplicable to his system. To force the concept of a "base" upon his system we must modify it somewhat. If each number of the two progressions of Napier is divided by 10^7, so that 0 becomes the logarithm of 1, then 1 is the logarithm of $(1 - 1/10^7)^{10^7}$, which is nearly equal to e^{-1}, where e is the base of the natural system. Hence the base of Napier's logarithms, when modified as here indicated, is very nearly e^{-1}.

It is well-known that Joost Bürgi invented logarithms independently of Napier, but he lost all rights of priority by failure to publish until the praises of Napier's book began to resound throughout Europe. In 1620 appeared in Prag the *Progress-Tabulen,* containing Bürgi's logarithmic tables, but omitting

[1] *The Construction of the wonderful Canon of Logarithms* by JOHN NAPIER. Translated from Latin into English by W. R. MACDONALD, Edinburgh and London, 1889, p. 19.

the explanations of them that were promised on the title-page.[1] Hence his logarithms were unintelligible to the ordinary reader. Common to Bürgi and Napier was the use of progressions in defining logarithms. In Bürgi's tables the numbers in the arithmetic progression were printed in *red*, the numbers in the geometric progression were in *black*. The relation between Bürgi's logarithms, $10n$, and their antilogarithms is expressed in modern notation by the equation $10n = \log [10^8(1 + 1/10^4)^n]$, $n = 1, 2, 3, \cdots$. The notion of a "base" can no more be forced upon Bürgi's logarithms than it can be upon the logarithms in Napier's tables.[2] In neither system is $\log 1 = 0$. Their logarithmic concepts were more general than those of the present day in this respect, that by sliding one progression past the other they could select any positive number at random as the one whose logarithm is zero. We have seen that Napier originally chose $\log 10^7 = 0$ while Bürgi chose $\log 10^8 = 0$. The logarithms in their tables were integral numbers. More than this, the terms of the two series could be made to increase in the same direction or in opposite directions, at pleasure. That is, if $m > n$, one can make $\log m < \log n$, or $\log m > \log n$, just as one may choose. Napier originally chose the first alternative, Bürgi the second.

It is generally known that Napier and Briggs conferred with each other and agreed to modify the original logarithms of Napier. In the *Appendix* to Napier's posthumous work, the *Constructio*, an improvement is suggested, "which adopts a cypher as the Logarithm of unity, and 10,000,000,000 as the Logarithm of either one tenth of unity or ten times unity." The subsequent use of decimal fractions in logarithmic tables led to the common logarithm proper, in which $\log 1 = 0$ and $\log 10 = 1$. A readjustment of Napier's original logarithms was made in John Speidell's *New Logarithmes*,[3] published in 1619 in London, whereby the logarithms virtually became the so-called "natural logarithms" of to-day.

Since at that time logarithms were studied in connection with some geometric progression having a positive initial term and a positive ratio between successive terms, so that negative terms could not arise in the progression, the question did not force itself for consideration, what the logarithm of a negative number should be. So far as we have noticed, no writer of the seventeenth century has raised this question. We shall see that Leibniz raised it in 1712. It would not be surprising had it arisen earlier. Charles Reyneau in his *Analyse démontrée*, Paris, 1708, Vol. II, p. 802, gives the formula for differentiation, "$dl - x = - 1dx/x$," but changes it in the table of errata to "$d - lx = - 1dx/x$." Did Reyneau fail to receive the suggestion from the printers' devil's "$l - x$"? In the seventeenth century the theory of logarithms was really on a very satisfactory

[1] These explanations were printed for the first time in *Grunert's Archiv*, Vol. 26, 1856, p. 323, in an article giving valuable historical notes relating to Bürgi.

[2] Misleading is the statement of M. Koppe: "... die Byrgschen [logarithmen] haben zur Basis $(1 + 10^{-4})^{10000}$, d. h. eine rationale Zahl, die mit e auf 4 Stellen übereinstimmt" (*Bibliotheca mathematica*, 3d S., Vol. 3, 1902, p. 151), as well as the statement of Felix Klein: "... dass die Bürgische Basis $(1{,}0001)^{10000} = 2{,}718146 \ldots$ mit e bereits auf 3 Dezimalen übereinstimmt" (F. Klein, *Elementarmathematik von höh. Standpunkte aus*, I, Leipzig, 1908, p. 333). From Bürgi's tables we copy $\log 134983856 = 30000$. It is readily seen that the base $e = 2.718 \ldots$ does not fit here.

[3] John Speidell's book is reprinted in Maseres' *Scriptores logarithmici*, Vol. 6, 1807, pp. 728–759.

basis. The definition of logarithms was restricted so as to apply only to positive numbers; to every positive number there corresponded one and only one logarithm. The needs of the practical computer were met. No necessity arose then to extend the logarithmic concept to negative or complex numbers.

A revision of Napier's definition of the logarithm of positive numbers was made by Edmund Halley. In 1695[1] he used these words: "The Old definition of Logarithms, that they are *Numerorum proportionalium æquidifferentes comites,* is too fancy to define them fully: For they may much more properly be said to be *Numeri Rationum Exponentes.*" "Thus, if there be supposed between 1 and 10 an infinite Scale of mean Proportionals, whose Number is 100,000, etc., *in infinitum;* between 1 and 2 there shall be 30102, etc., of such Proportionals, and between 1 and 3 there will be 47712, etc., of them, which Numbers therefore are the Logarithms of the Rationes, of 1 to 10, 1 to 2, and 1 to 3; and not so properly to be called the Logarithms of 10, 2, and 3." Halley's article, from which we quote, is obscurely and inaccurately worded, but possesses the merit of giving the first derivation, by processes divorced from geometry (the hyperbola), of infinite series for the computation of logarithms.

Halley's idea of logarithms of "ratios" involves an interesting point which historians have hitherto completely overlooked. Halley anticipated Roger Cotes by nineteen years in the introduction of a certain mode of measuring a ratio. Halley considers ratio a "*quantitas sui generis.*" Yet Halley's idea finds its inception in Napier's ratios of distances of two flowing points. Halley says: "These Rationes we suppose to be measured by the Number of Ratiunculae contained in each." This number stands for the logarithm of the ratio. But "logarithms thus produced may be of as many forms as you please." If in place of 100,000 mean proportionals between 1 and 10 we take 230258 of them, then, instead of common logarithms, we get the natural logarithms. Every system of logarithms differs by a constant factor from some system chosen as a standard. Hence, for a given ratio, the number of ratiunculæ is arbitrary; it is equal to the number in the standard system, multiplied by a constant. In other words, *the measure of a ratio is a constant times the standard logarithm of that ratio.* This is the idea, here expressed by Halley, later elaborated by Cotes, then largely lost sight of until the nineteenth century, when F. Klein introduced it anew, in order that he might establish the validity of the processes of projective geometry for the hypotheses of non-euclidean geometry.[2]

The following quotation from the article "Logarithms" in the second edition (1743) of E. Stone's *New Mathematical Dictionary* is of interest, because it asserts the dependence of Cotes upon Halley and especially because it gives utterance to a feeling of strangeness toward Halley's definition of the "measure of a ratio"

[1] *Philosophical Transactions,* London, No. 216, p. 58.

[2] Klein, *Götting. Nachrichten,* 1871, No. 17, and *Mathem. Annalen,* Vol. 4, 1871, pp. 573–625. It is possible that even before Halley this idea of the measure of a ratio may have been entertained by Kepler, for he speaks of "*proportio, ejusque mensura, logarithmus.*" See Kepler, *Tabulæ Rudolphinæ,* Ulm, 1627, Cap. III, p. 11. In fact, Charles Hutton, in his *Philosophical and Mathematical Dictionary,* London, 1815, Art. "Exponent," mentions "the idea of measures of ratios, as delivered by Kepler, Mercator, Halley, Cotes, etc."

which seizes one who has not gone beyond the elementary concepts of measurement. Stone says:

> "Mr. Cotes too, at the Beginning of his *Harmon. Mensur.* has done this Business in imitation of Dr. Halley, altho' more short, yet with the same Obscurity: for I appeal to any one, even of his greatest Admirers, if they know what he would be at in his first Problem, *viz. to find the Measure of a Ratio* from the Terms of the Problem itself. . . ."

"Logarithms the measures of ratios" is referred to by Saunderson in his *Algebra*,[1] and by W. J. G. Karsten of Halle, in an important paper on logarithms,[2] and by J. F. Lorenz, in his *Elemente der Mathematik*,[3] but in general, this definition failed to influence mathematical thought of the eighteenth century.

The early explanations of the use of logarithms involved the consideration of many special cases. The theorems $\log a + \log b = \log ab$, $\log a - \log b = \log a/b$, $\log a^m = m \log a$, were explicitly stated by Oughtred (not in algebraic symbols, as here, but in words) in his booklet *De æquationum affectarum resolutione in numeris*, which was published in 1652, bound in one volume with his *Clavis mathematica.* The 1631 edition of the *Clavis* does not give these theorems.[4]

The theoretic viewpoint of the logarithm was broadened somewhat during the seventeenth century by the graphic representation, both in rectangular and polar coördinates, of a variable number and its variable logarithm. Thus were invented the *logarithmic curve* and the *logarithmic spiral.* The graphic representation of functions was then just beginning to be understood. Probably the logarithmic curve suggested itself to many minds. It has not been possible thus far to ascertain with certainty who was the first inventor of it. Hutton says[5] that the curve has been attributed to Edmund Gunter, but Hutton could not find it in Gunter's writings. Most likely the rumor is due to a confusion between the "logarithmic curve," and the "logarithmic line" invented by Gunter and known as the forerunner of the slide rule. It has been thought that the earliest reference to the logarithmic curve was made by the Italian Evangelista Torricelli in a letter of the year 1644,[6] but Paul Tannery has made it practically certain that Descartes knew the curve in 1639.[7]

It is found in a work of J. Gregory of the year 1667[8] and is clearly explained in the second edition (1690) of a book by the French mathematician, C. F. M. Dechales. In the same year Christiaan Huygens made known without proof the beautiful properties of the logarithmic curve;[9] these properties were proved later

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[1] *Select Parts of Professor Saunderson's Elements of Algebra*, 3d ed., London, 1771, p. 402.

[2] *Abh. Münch. Acad.*, V, 1768.

[3] J. F. Lorenz, *Elemente*, I Theil, Leipzig, 1793, p. 140.

[4] H. Bosmans, "La première édition de la Clavis mathematica d'Oughtred," *Annales de la Société scient. de Bruxelles*, T. 35, 2. partie, p. 38.

[5] C. Hutton, *Math. Tables*, London, 1811, Introduction, p. 84.

[6] G. Loria in *Bibliotheca mathematica*, 3d S., Vol. I, 1900, p. 75.

[7] *L'intermédiaire des mathématiciens*, T. VII, 1900, p. 95. Tannery refers to a letter of Descartes, dated Feb. 20, 1639 (Édit. of Descartes, 1677, Vol. III, letter 71).

[8] J. Gregory, *Geometriæ pars universalis*, Venetiæ, 1667; See Montucla, *Histoire des mathématiques*, II, 1799, p. 85.

[9] C. Huygens, *De la cause de la pesanteur*, published in 1690 as an appendix to the *Traité de la lumière*.

by G. Grandi[1] and G. Fontana.[2] Theorems on the quadrature of this curve were given by Torricelli, Huygens and J. Craig.[3] This curve was discussed also by J. Bernoulli.[4] Its rectification was first explained in 1692 by Marquis l'Hospital in a letter to Leibniz.[5] We shall see that during the eighteenth century the logarithmic curve plays a leading rôle in the discussion of the theory of logarithms, and that it helped to cloud rather than clarify the questions at issue.

The curve which is the graphic representation in polar coordinates of the relation between a variable and its logarithm was invented by René Descartes. It is described by him in 1638 in a letter to P. Mersenne.[6] Descartes does not give its equation, nor does he connect it with logarithms. He describes it as the curve which makes equal angles with all the radii drawn through the origin. Soon after, this spiral was re-invented by Torricelli, who, as we have seen, is second only to Descartes in being unquestionably associated with the history of the logarithmic curve. The name, *logarithmic spiral,* was coined by Pierre Varignon in a paper presented to the Paris academy in 1704 and published in 1722.[7] In England this spiral commanded the attention of Oughtred, Collins, Wallis and Barrow.[8]

No less important in the history of logarithms is a third curve, namely the hyperbola. The quadrature of the space between the hyperbola and its asymptotes was effected by Gregory St. Vincent in Book VII of his *Opus geometricum,* Antwerp, 1647.[9]

This area, for the rectangular hyperbola $xy = 1$, is expressed now in the logarithmic form log y_1/y, the area thus indicated being bounded by the axis of x, the ordinates y and y_1, and the hyperbolic arc. Nevertheless, this investigation of Gregory St. Vincent, strictly speaking, does not figure in the history of logarithms. He does not mention logarithms. His result is purely geometric and would remain unaltered in every particular, had logarithms never been invented. What he established was simply the theorem that, if parallels to one asymptote are drawn between the hyperbola and the other asymptote, so that the successive areas of the mixtilinear quadrilaterals thus formed are equal, then the lengths of these parallels form a geometric progression.[10]

Apparently the first writer to state this theorem in the language of logarithms

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[1] *Geometrica demonstratio theorematum Hugeniorum circa logisticam seu logarithmicam,* Florenz, 1701.

[2] *Sopra il centro di gravità della logistica finita ed infinitamente lunga,* Torino, Mem. X and XI. See also Loria, *Ebene Kurven,* Deutsche Ausgabe von F. Schütte, Leipzig, 1902, p. 543.

[3] *The quadrature of the logarithmic curve* in *Philosoph. Transactions,* No. 242, 1698.

[4] *Acta Eruditorum,* 1696, p. 216.

[5] Leibniz, *Werke,* Ed. Gerhardt, Vol. II, p. 216.

[6] *Œuvres de Descartes,* éd. Cousin, VII, Paris, 1824, pp. 336–37; Éd. Adam et Tannery, II, Paris, 1898, p. 360. See also G. Loria, *Ebene Kurven,* Leipzig, 1902, pp. 448–456.

[7] Loria, *op. cit.,* p. 444.

[8] A. Favaro, *Bibliotheca Mathematica,* N. F., 5, 1891, pp. 23–25.

[9] Karl Bopp, "Die Kegelschnitte d. Gregorius a St. Vincentio," Leipzig, 1907, in *Abhandl. z. Gesch. d. math. Wiss.* (M. Cantor), XX Heft.

[10] K. Bopp, *op. cit.,* p. 265, Propos. CXXX in book on Hyperbola. A different mode of statement is given in Propos. CXXV.

was the Belgian Jesuit Alfons Anton de Sarasa,[1] who defended Gregory St. Vincent against attacks made by Mersenne. This statement was a very natural step to take, in view of Gregory St. Vincent's theorem that, in a hyperbola $xy = 1$, the asymptotic area varies in arithmetic ratio when y or x varies in geometric ratio, and of Napier's definition of a logarithm which established a one-to-one correspondence between a geometric series and an arithmetic series.

An important publication was Nicolaus Mercator's booklet, the *Logarithmotechnia*, London, 1668. Mercator writes the equation of the hyperbola in the form $y = 1/(1 + a)$, where $1 + a$ is the abscissa, and y the ordinate. He expands $1/(1 + a)$ by division into an infinite series (in itself a novel undertaking),

$$\frac{1}{1+a} = 1 - a + aa - a^3 + \cdots.$$

He gives a crude explanation of the process of summation which we now indicate by $\int x^n dx = x^{n+1}/(n + 1)$, and then integrates the terms of the above series. One would expect him to write down the logarithmic series $\log(1 + a) = a - aa/2 + a^3/3 - \cdots$, which is attributed to him, but he does not do so. Instead of this he writes down the numerical values of the first few terms of that series, taking $a = .1$, thereby obtaining the area of the mixtilinear quadrilateral between the ordinates $y = 1$ and $y = 1 \div 1.1$ as .095310181. He repeats this process for $a = .21$.[2] Probably using the results reached by Gregory St. Vincent and de Sarasa, Mercator finally connects his own results with logarithms. That he really used the logarithmic series is evident also from the next step, which must have been obtained by integration of the terms of this series. He does not write down the general result of this integration, but again gives only the numerical values corresponding to the first few terms of the new series, for $a = .1$. Thus he obtains the value of $\int \log(1 + a)da$, for $a = .1$. These results are found at the very end of Mercator's booklet, under the caption, *Invenire summam logarithmorum*. Tropfke is of the opinion that Mercator's failure to print the general logarithmic series was due to a practice still prevalent in his day of merely hinting at new results, in order to maintain an advantage over others.[3] We do not consider this the real motive. Any reader could repeat Mercator's areal computations from the full explanations given. His course is probably due to a different conception of the proper style of exposition, which favored the concrete special case to the general formula. Wallis was the first to state Mercator's logarithmic series in general symbols.[4] While the investigations due to Gregory St. Vincent, Mercator and some others led to greatly improved methods of computing logarithms by infinite series, and to the phrasing of geometric theorems

[1] *Solutio Problematis a R. P. Marino Mersenno propositi*, 1649. See Cantor, *op. cit.*, Vol. II, 2d ed., pp. 714, 715.

[2] See Maseres, *Scriptores logarithmici*, Vol. I, London, 1791, p. 194. Maseres reprints here the entire *Logarithmo-technia*.

[3] Tropfke, *op. cit.*, II, 1903, p. 182.

[4] Maseres, *op. cit.*, Vol. I, p. 229; *Phil. Trans.*, No. 38, year 1668.

in the language of logarithms, no modification of the logarithmic concept resulted from these researches.

In a MS. of July, 1676, Leibniz[1] derives the integral $\int dy/y$. It arose in the study of the problem which Florimond de Beaune had propounded to Descartes who replied in a letter of Feb. 20, 1639:[2] To find the quadrature of that curve in which the ordinate is to the subtangent as a given line segment is to the difference of the ordinate and abscissa. In this MS. of 1676, as well as in the published paper, the *Nova methodus*,[3] Leibniz is led by this problem to the equation $w/a = dw/dx$. He takes dx as a constant b, and gets $w = a/b \cdot dw$; that is, the ordinates w are proportional to their increments. If the x's increase in arithmetic progression, then the corresponding w's are the terms of a geometric progression. Hence the x's are the logarithms of the w's; he concluded that the curve is logarithmic.[4] As early as 1675 Leibniz used the notation "Log y," as in the equation "$a^2 - yx = 2y^2 \text{ Log } y$."[5]

The Modern Exponential Notation.

The modern symbolism for powers of numbers was introduced by René Descartes in his *La géométrie*, Paris, 1637. He writes "*aa* ou a^2 pour multiplier a par soimême; et a^3 pour le multiplier encore une fois par a, et ainsi à l'infini."[6] Thus, while Vieta represented A^3 by "*A* cubus" and Stevin x^3 by a figure 3 within a small circle, Descartes wrote a^3. In his *Géométrie* he does not use negative and fractional exponents, nor literal exponents. His notation was the outgrowth and an improvement of notations employed before him by Chuquet, Bombelli, Bürgi, Reymer, and Kepler. Chuquet's manuscript work, *Le Triparty en la science des nombres*, 1484, gives $12x^3$ and $10x^5$, and their product $120x^8$, by the symbols 12^3, 10^5, 120^8, respectively. Chuquet goes even further and writes[7] $12x^0$ and $7x^{-1}$ thus 12^0, 7^{1m}. He represents the product of $8x^3$ and $7x^{-1}$ by 56^2. Bürgi, Reymer and Kepler use Roman numerals for the exponential symbol. Bürgi writes $16x^2$ thus $\overset{\text{II}}{16}$. Thomas Harriot simply repeats the letters; he writes[8] $a^4 - 1024a^2 + 6254a$ thus: $aaaa - 1024aa + 6254a$.

Descartes's notation spread rapidly. It was used by Fr. v. Schooten in his commentary on Descartes's *Géométrie*, in the edition which appeared in Amsterdam in 1644.[9] This notation was used by Huygens and Mersenne in 1646 in their correspondence with each other,[10] and by Hudde in 1658.[11] Oughtred did

[1] C. I. Gerhardt, *Entdeckung der Differentialrechnung durch Leibniz*, Halle, 1848, pp. 53, 54.
[2] Cantor, *Geschichte der Mathematik*, Vol. II, 1892, p. 781.
[3] *Acta Eruditorum*, 1684 = Leibniz, *Werke*, Vol. V, 1858, pp. 220–226.
[4] Cantor, *op. cit.*, III, 1898; pp. 187, 188.
[5] Gerhardt, *op. cit.*, p. 35.
[6] *La géométrie de René Descartes*. Nouvelle Édition. Paris, 1886, p. 2.
[7] Chuquet, "Le triparty," *Bullettino Boncompagni*, Tomo XIII, Rome, 1880, p. 740.
[8] Harriot, *Artis analyticæ praxis*, London, 1631, p. 156.
[9] Matthiessen, *Grundzüge der Antiken u. Modernen Algebra*, Leipzig, 1878, p. 551.
[10] C. Huygens, *Œuvres*, T. I, La Haye, 1888, p. 24.
[11] Joh. Huddenii *Epist. I de reductione æquationum*, Amsterdam, 1658; Matthiessen, *op. cit.*, p. 374.

not use the modern exponents in any of his editions of his *Clavis mathematica* (London, 1631, 1648, 1652), but if Rigaud reproduced faithfully the notations in the original letters, it follows that Oughtred used positive integral exponents in his correspondence as early as 1642.[1] On Feb. 5, 1666–7, John Wallis wrote to John Collins, a proposed new edition of Oughtred's *Clavis* being under discussion: "It is true, that as in other things so in mathematics, fashions will daily alter, and that which Mr. Oughtred designed by great letters may now by others be designed by small; but a mathematician will, with the same ease and advantage, understand Ac and a^3 or aaa."[2] John Pell wrote r^2 and t^2 in a letter written in Amsterdam on Aug. 7, 1645.[3] Pascal made free use of positive integral exponents in several of his papers, particularly the *Potestatum numericarum summa*, 1654. G. Kinckhuysen used positive integral exponents in 1660, in his *Meet-Konst,* and in 1661 in his *Algebra.*[4]

[The February issue will conclude the discussion of the modern exponential notation and take up the unsuccessful attempts to create a theory of logarithms of negative numbers.]

ERRORS IN THE LITERATURE ON GROUPS OF FINITE ORDER.

By G. A. MILLER, University of Illinois.

Errors in General.

Although mathematics is an exact science, the mathematical literature is disfigured by numerous errors. Some of the most influential works fail along this line. In the preface to volume 1 of his *Zahlentheorie,* 1892, Paul Bachmann observes that Gauss' *Disquisitiones arithmeticæ,* 1801, was scarcely readable on account of the large number of annoying typographical errors.

Imperfect Proofs.—While typographical errors are annoying they are not the most serious errors. A much more annoying class of errors is due to imperfect proofs. The careless use of infinite series during the seventeenth and eighteenth centuries, without any inquiry as to convergence, is one of the most important instances of imperfect methods. In fact, in the early part of the nineteenth century there were only a few mathematicians who were sufficiently careful about this matter; so that Abel could properly write, in 1826, to his friend Holmboe, that most of the papers dealing with series were inexact.[5] Recently Study called atten-

[1] S. J. Rigaud, *Correspondence of Scientific Men of the Seventeenth Century,* Vol. I, Oxford, 1841, p. 63.

[2] Rigaud, *op. cit.,* Vol. I, p. 475.

[3] J. O. Halliwell, *Progress of Science in England,* London, 1841, p. 89.

[4] Kinckhuysen, *De Grondt der Meet-Konst,* De Haerlem, 1660; *Algebra ofte Stel-Konst,* De Haerlem, 1661.

[5] Niels Henrik Abel Memorial, Correspondance, 1902, p. 16.

tion to the great lack of precision in much of our recent geometrical production, referring to it as "indegesta moles geometrica."[1]

A third large class of errors is due to the oversight of an important step in an otherwise perfect proof. Many of the supposed proofs of the parallel axiom, starting with those of Ptolomy and Proclus, are of this nature. This is also true of many of the supposed proofs of the fundamental theorem of algebra, including those associated with such prominent names as d'Alembert, Euler, Lagrange, and Laplace.[2] Many recent examples of such errors are found in the literature relating to attempts to prove the great Fermat theorem.

Historical References.—Another large class of errors relates to historical notes. As a result of the recent very rapid progress in the development of our knowledge of the history of mathematics a very large number of the older mathematical references have been proved to be incorrect. Even some of the most recent and most reliable books disclose a lack of sufficient care in this respect. As instances of this kind, we may mention the erroneous statements regarding the early use of the term function, given both in the *Fundamental Concepts of Algebra and Geometry* by J. W. Young, 1911, page 194, and also in the *Monographs on Modern Mathematics* by J. W. A. Young, 1911, page 265.[3] It is, of course, generally unnecessary to give historical references in a mathematical work, but when they are given they should be correct.

Lists of Errors.—In 1904 Maillet proposed, as question number 2855 in *L'Intermédiaire des Mathématiciens*, the making of a list of errors committed by prominent deceased mathematicians. Maillet began this list by referring to errors committed by Ampère, Abel, Chasles, Legendre, Bertrand, Laplace, and others. Many others have contributed to this list but it is still very incomplete. It is, however, sufficiently extensive to exhibit many interesting questions with important caution signals, and it is to be hoped that this list will be gradually extended. A similar list relating to the errors committed by the great living mathematicians would probably be still more useful.

The question of errors in mathematical literature is especially important at the present time in view of their rapid increase on account of the numerous efforts to prove Fermat's theorem, and in view of the great encyclopedias of mathematics which are in course of publication. Both of these encyclopedias invite the mathematical world to coöperate in the effort to make advances and to banish falsehoods. For this purpose the German encyclopedia established a "Sprechsaal" in the *Archiv der Mathematik und Physik* which was transferred in 1907 to the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, while the French edition publishes a "Tribune publique" in connection with the regular numbers of the encyclopedia. It is expected to embody all the valuable features of the "Tribune publique" in the "Additions et Modifications" which are to appear near the end of each volume. In view of the extensive and permanent circulation

[1] Study, *Archiv der Mathematik und Physik*, volume 18 (1911), p. 175.

[2] Cf. *Encyclopédie des Sciences Mathématiques*, tome 1, vol. 2, p. 193.

[3] This error seems to have been started by d'Alembert. It has been corrected in the *Encyclopédie des Sciences Mathématiques*, tome II, vol. 1, p. 3.

of these encyclopedias it is extremely important that they should be as reliable as possible, and it is to be hoped that their appeal for public coöperation will lead to more and more valuable responses. It is almost as important that matters of no value be excluded from such standard works as it is that all matters which exhibit real progress should be included.

First Aid to Truth.—For the first correction of historical errors the *Bibliotheca Mathematica* affords an excellent medium. This journal has already rendered most valuable services by directing attention to the large number of errors in some of the most popular works on the history of mathematics, and by exposing these errors in a fearless manner. It should be remembered that errors are weeds and that they tend to impede the growth of the truth. It is one of the main duties of the true scholar to assist in their destruction, since this is essential for a perfect harvest. The man who fearlessly and unselfishly exposes errors, especially when they appear in the best books, is thereby rendering a great service, which is frequently not sufficiently appreciated.

Reviews constitute the most important single instrument to combat errors. Unfortunately reviewers do not always use this instrument wisely. In many cases this may be due to a lack of self-confidence on the part of the reviewer. In other cases it is doubtless due to the inclination to write a review which may please the author of the work under review. A conscientious reviewer cannot escape the sense of duty to the public and to truth. If we could have more reviews like those which Study, of Bonn, writes, we should doubtless have a more enlightened public and a larger number of careful authors.

Students of mathematics should read more semi-mathematical literature, such as these reviews, and the historical or philosophical mathematical literature. Many new viewpoints come to us in this way. Errors due to eminent mathematicians furnish an especially valuable topic of a semi-mathematical nature, and even if a student reads reviews of books and of articles mainly from a sense of curiosity as to how the work of those whom he knows is regarded by competent critics, it is very much better than if he reads nothing outside of those things which bear directly on his regular work. It should be remembered, however, that it is very easy to discover little errors in a great man's work, just as it is easy to make slight extensions to such work. The correction of really big and important errors is the work of a big man. A man discloses his caliber by the way he corrects errors just as much as by his other publications.

Errors in the Early Literature of Group Theory.

Galois was the first to enter deeply into the theory of groups. In his efforts to develop a general theory of equations he directed attention to the important concept of simple groups and proved that the icosahedral group is the smallest simple group of composite order. He also studied a very important infinite system of simple groups, known as the general modular groups, of order $\frac{1}{2}p(p^2 - 1)$ and of degree $p + 1$, $p > 3$. In one of his earliest papers, published in the

Bulletin des Sciences Mathématiques de M. Férussac, 1830, page 271, he asserts that this modular group can be represented as a substitution group of degree p only when $p = 5$. Later he corrected his mistake by observing that this representation is also possible when $p = 7$ or 11, but for no larger value of p. A recent proof of this important theorem is due to Frobenius, who also observes that the modular group is the only simple group of order $\frac{1}{2}p(p^2 - 1)$,[1] $p > 3$.

Errors of Cauchy.—Cauchy was the first who wrote extensively on the theory of substitution groups, and he is commonly called the founder of this theory in view of his many valuable contributions along this line. Unfortunately, the number of errors in his writings is very large, and most of them are repeated in the published parts of his complete works. As these works are brought out under the auspices of the Paris Academy of Sciences, one might have expected footnotes calling attention to the errors which are so distracting and which impair so seriously the usefulness of the work; especially since the publication is proceeding very slowly, having commenced about thirty years ago. Not only do these writings involve many erroneous statements but they also contain obscurities which later developments have completely cleared away. Those who edit collected works do not always take sufficient care to avoid perpetuating the errors and obscurities of a writer.

Cauchy began the construction of fairly long lists of substitution groups which are supposed to contain all the possible substitution groups on six or a smaller number of letters. Such lists were based by Cauchy upon an insufficient knowledge of the theory of substitutions and hence they were incomplete. Very many later writers had a similar experience and the efforts to enumerate all the possible substitutions groups of certain degrees have led to a very large number of errors in the literature of group theory, especially on the part of Kirkman, Askwith, and Cayley. One of the most important instances of false theorems announced by Cauchy is the one which states that every simply transitive group of degree $p + 1$, p being a prime number, is imprimitive.[2]

Errors of Cayley.—Cayley's writings on group theory are scarcely more accurate than those of Cauchy. In volume 1 of the *American Journal of Mathematics*, page 51, Cayley states that there are three groups of order 6, giving as examples, the non-cyclic group and the cyclic group in two different forms. It seems very strange that Cayley should have made a mistake regarding such a simple matter more than twenty years after he began to study group theory. About thirteen years later Cayley attempted to enumerate the substitution groups[3] whose degrees do not exceed 8 and in this work the number of errors is very large, as has been observed by various writers. These errors have been repeated in the *Collected Mathematical Papers*, volume XIII, 1897, and the very evident error in reference to the groups of order 6 is also repeated in volume X, 1896.

Errors of Jordan.—Jordan did very much pioneer work in group theory and he attacked some of the most difficult problems. While the number of errors in

[1] Frobenius, *Berliner Sitzungsberichte*, 1902, page 351.
[2] Cauchy, Paris, *Comptes Rendus*, volume 21 (1845), page 1199.
[3] These enumerations appeared in *Quarterly Journal of Mathematics*, volume 25, 1891.

his works is considerable it is perhaps not larger than one could reasonably expect from the difficulty and newness of the problems. In particular, as early as 1878 he made the first attempt to determine all the finite collineation groups in three variables, but he failed to find the two important groups of orders 168 and 360, which were discovered later by Klein and Valentiner respectively. He also attempted to determine all the primitive substitution groups which can be represented on no more than 17 letters, but in this enumeration he also omitted a number of important groups. In particular he found only 14 out of 22 primitive groups of degree 16.

Errors in Definitions.—The fact that early writers generally did not use a precise definition of group is a source of considerable confusion.[1] Even in such a good work as Klein's *Ikosaeder,* page 5, we read that a set of operations constitute a group when the composition of two of them gives one of the set. It is, of course, true that such a general definition may be used, and one finds authority for it in the French encyclopedia, tome 1, volume 2, page 243. In fact, the term group is defined in two very distinct ways in this standard work of reference. A commonly accepted definition is found on page 576 of the first volume. It seems to the writer unfortunate that two distinct definitions of this widely used term should appear in a work of reference, since this may tend to perpetuate its use in a vague manner by careless writers.

A Few Recent Errors in Group Theory.

Errors in the Encyclopædias.—A queer historical error occurs under the word "group" in the latest edition of the *Encyclopædia Britannica.* It is stated here that the mathematical technical sense of the word group is not older than 1870. If we recall that Cayley used the term group with its technical meaning in the heading of a number of his articles, beginning as early as 1854; and that Kirkman and Sylvester also used it about 1860, it appears very strange that an Englishman should have made the above statement. Numerous other errors relating to the history of mathematics occur in the latest edition of this popular work of reference but none of them appears less excusable than the one which has just been noted.

That the great French mathematical encyclopedia is not free from errors in its article on group theory may readily be seen from the many corrections which have appeared in the "Tribune publique." The German mathematical encyclopedia devoted such a small amount of space to the subject of finite groups that one could scarcely expect many errors. There are, however, also several somewhat serious errors in this work. Perhaps the most important one occurs on page 221, where it is stated, in substance, that the holomorph of a group is its group of isomorphisms, although the term holomorph is not used.

A very important recent general article on groups of finite order appeared in the latest edition of *Pascal's Repertorium.* This article is also unusually

[1] The present abstract definition is due to Germans; especially to Kronecker (1870), Weber (1882), and Frobenius (1887).

free from errors, notwithstanding its broad scope. One of the most important errors in it occurs on page 229, where the fundamental theorem, that with each regular group there may be associated a similar regular group in the same letters such that each of these groups is composed of all the substitutions in these letters which are commutative with every substitution of the other, is credited to Cayley (1878). As a matter of fact Jordan proved this theorem in his thesis published in 1860 and republished in *L'École Polytechnique* during the following year.

Error of Weber.—A very interesting error occurs in volume 2, page 54, of the first edition of Weber's *Lehrbuch der Algebra.* It is stated here that the most important example of a commutative group is furnished by the natural numbers when they are combined by ordinary multiplication. As a matter of fact these numbers do not form a group at all, as Weber himself observes in the second edition of his classic algebra. This most important example of a commutative group is therefore no group. In this connection we may refer to a historic misstatement made by Weber several years ago and repeated by many others. In his biography of Kronecker, which appeared at the beginning of volume 43 of the *Mathematische Annalen,* Weber states that Kronecker mentions Galois for the first time in a letter to Dirichlet in 1856. On the other hand, Kronecker had referred several times to Galois' theory in an article which was read several years earlier and appeared in the *Berichte* of the Berlin Academy for 1853.

Inconsistent Definitions.—An instance where the technical and the non-technical meaning of the word group are confused may deserve notice here, since it occurs in a very useful reference work; namely, in volume I of the *Subject Index* of the Royal Society of London Catalogue of Scientific Papers, which was published in 1908. On page 66 of this index we find a reference to Cockle's papers on "Vanishing Groups." This reference appears under the technical subject groups, but the papers by Cockle refer entirely to groups in the non-technical sense and have therefore nothing to do with groups as a technical concept.

The fact that the same name is assigned to two distinct groups in two well-known works of reference is a source of some confusion. In volume I, page 238, of the latest edition (1910) of *Pascal's Repertorium,* the symmetric group of order 6 is called the *anharmonic group,* while Capelli uses the same name for an entirely different group in the most recent edition of his algebra, entitled, *Istituzioni di analisi algebrica,* 1909, page 111. It may also be observed that some writers use the term degree for what is commonly known as the order of a group, and that the term metacyclic is used with different meanings.

Lack of References.—The literature of group theory has been made unnecessarily complicated by a lack of a proper appreciation of the importance of due references. In fact, some writers have not only failed to give proper references but have not even taken the trouble to find out whether their papers contained any new results before offering them for publication. In many cases such errors are not noted even in the reviews which appear in such good works as the *Jahrbuch ueber die Fortschritte der Mathematik.*

As an instance of this kind we may cite an article by Netto which appeared in *Crelle's Journal*, volume 128 (1905), page 243. We find in this article a development of the tetrahedral group according to an abstract definition which has been well known for more than half a century, and yet no references are given by the author of the article in question or by the reviewer in the *Fortschritte*. The article contains a number of other things which had been published several years before this article appeared. An instance of an article which contained nothing new and yet appeared in the excellent *Mathematische Annalen,* without proper reference, may be found by consulting this journal, volume 60 (1905), page 319.

Errors of Judgment.—There is a class of indirect errors to which it may be desirable to refer here; namely, those which arise from undue emphasis on some particular phase of the subject or on the work of some particular man. According to the writer's opinion Burnside's *Theory of Groups* errs along this line by quoting Hölder's work relatively too frequently. In fact, these references are not always correct. Even in the second edition, page 39, we are told that the symbol for quotient group was introduced by Hölder, although this symbol had been employed by Jordan at a much earlier date. As other instances of undue emphasis in important publications with respect to the work of one man, we may cite the references to Frobenius in the excellent article on groups contained in *Pascal's Repertorium* and the references to the present writer in the French encyclopedia article on this subject.

Conclusion.—It may be added that the above are only a few typical examples of the many errors which have been committed by some of those who have contributed much to the development of group theory. A complete collection of such errors, with proper explanations, would make a volume. Some of these errors have not had an unmitigatedly baneful effect, as they inspired young investigators with confidence, when these found that they could correct mistakes in the works of eminent mathematicians. The contributions to mathematics, due to the inspiration received from the discovery of errors in the works of eminent men, have been of great value to our science; but the beauty and insight furnished by correct and fundamental results are still more inspiring and are much more worthy objects of our endeavors.

ON THE REMAINDER TERM IN A CERTAIN DEVELOPMENT OF $f(a+x)$.*

By R. D. CARMICHAEL, Indiana University.

In the *Annals of Mathematics*, Vol. 5, No. 4, July, 1904, Mr. S. A. Corey gave an interesting and important development of $f(a+x)$, and showed that his formula was admirably adapted to the numerical computation of complicated

* Read before the American Mathematical Society, February 29, 1908.

definite integrals. The object of this paper is to find the "remainder term" in Corey's series* and incidentally to verify the series.

Putting the expansion in a form somewhat different from that of Corey and writing after it the remainder term R_{2n+2}, we shall verify the series and find the value† of R_{2n+2}:

$$\begin{aligned} f(a+x) = f(a) &+ \frac{x}{m}\left[f'\left(a+\frac{x}{m}\right)+f'\left(a+\frac{2x}{m}\right)+\cdots+f'\left(a+\frac{m-1}{m}x\right)\right. \\ &\left.+f'(a+x)\right] - \frac{x}{m\cdot 2}[f'(a+x)-f'(a)] - \frac{B_1x^2}{m^2\cdot 2!}[f''(a+x)-f''(a)] \\ &+ \frac{B_2x^4}{m^4\cdot 4!}[f^{(\mathrm{iv})}(a+x)-f^{(\mathrm{iv})}(a)] - \cdots \\ &+ (-1)^n\frac{B_nx^{2n}}{m^{2n}\cdot(2n)!}[f^{(2n)}(a+x)-f^{(2n)}(a)] + R_{2n+2}. \end{aligned} \tag{1}$$

Here m is a positive integer and $B_1, B_2, \cdots$ are the well-known Bernoulli numbers:

$$B_1 = \tfrac{1}{6}, \quad B_2 = \tfrac{1}{30}, \quad B_3 = \tfrac{1}{42}, \quad B_4 = \tfrac{1}{30}, \quad B_5 = \tfrac{5}{66}, \quad \cdots.$$

Throughout the discussion it is assumed that $f(x)$ and its derivatives employed exist and are continuous over the range a to $a + x$.

When $r \leqq m$ one sees from Taylor's development with remainder term that

$$\begin{aligned} f'\left(a+\frac{rx}{m}\right) &= f'(a) + \frac{rx}{m}f''(a) + \frac{r^2x^2}{2!\cdot m^2}f'''(a) + \cdots \\ &+ \frac{r^{2n}x^{2n}}{m^{2n}\cdot(2n)!}f^{(2n+1)}(a) + \frac{r^{2n+1}x^{2n+1}}{m^{2n+1}\cdot(2n+1)!}f^{(2n+2)}(a+\varphi_rx), \end{aligned} \tag{2}$$

where $0 < \varphi_r < 1$. If in this equation r is put successively equal to $1, 2, 3, \cdots, m$, and the resulting equation in each case is multiplied by x/m, we have by addition:

$$\begin{aligned} &\frac{x}{m}\left[f'\left(a+\frac{x}{m}\right)+f'\left(a+\frac{2x}{m}\right)+\cdots+f'\left(a+\frac{m-1}{m}x\right)+f'(a+x)\right] \\ &= xf'(a)+x^2f''(a)\left[\frac{1+2+3+\cdots+m}{m^2}\right]+\frac{x^3f'''(a)}{2!}\left[\frac{1^2+2^2+3^2+\cdots+m^2}{m^3}\right] \\ &\quad + \cdots + \frac{x^{2n+1}f^{(2n+1)}(a)}{(2n)!}\left[\frac{1^{2n}+2^{2n}+3^{2n}+\cdots+m^{2n}}{m^{2n+1}}\right] \\ &\quad + \frac{x^{2n+2}}{m^{2n+2}\cdot(2n+1)!}[f^{(2n+2)}(a+\varphi_1x)+2^{2n+1}f(a+\varphi_2x)+\cdots \\ &\quad + m^{2n+1}f^{(2n+2)}(a+\varphi_mx)]. \end{aligned} \tag{3}$$

* Mr. Corey gave a partial solution of this problem in the AMERICAN MATHEMATICAL MONTHLY, Vol. 14, Nos. 6–7, pp. 131–135; June–July, 1907.

† The value in final form is given in equation (13) below.

Again from Taylor's development it follows that

$$
(4)\quad \begin{aligned} f^{(s)}(a+x) - f^{(s)}(a) &= xf^{(s+1)}(a) + \frac{x^2}{2!}f^{(s+2)}(a) + \cdots \\ &+ \frac{x^{2n-s+1}}{(2n-s+1)!}f^{(2n+1)}(a) + \frac{x^{2n-s+2}}{(2n-s+2)!}f^{(2n+2)}(a+\theta_s x), \end{aligned}
$$

where $0 < \theta_s < 1$. We shall again form a series of equations and take their sum. Let $s = 1$ and multiply the resulting equation by $-x/m\cdot 2$. Then take $s = 2$ and multiply the resulting equation by $-B_1x^2/m^2\cdot 2!$. Then take $s = 4$ and multiply by $B_2x^4/m^4\cdot 4!$, and so on. By addition we should finally have the following:

$$
\begin{aligned}
&-\frac{x}{m\cdot 2}[f'(a+x) - f'(a)] - \frac{B_1x^2}{m^2\cdot 2!}[f''(a+x) - f''(a)] + \frac{B_2x^4}{m^4\cdot 4!} \\
&\times [f^{(\text{iv})}(a+x) - f^{(\text{iv})}(a)] - \cdots + (-1)^n\frac{B_nx^{2n}}{m^{2n}\cdot(2n)!}[f^{(2n)}(a+x) - f^{(2n)}(a)]
\end{aligned}
$$

$$
(5)\quad \begin{aligned}
&= x^2f''(a)\left[-\frac{1}{m\cdot 2}\right] + x^3f'''(a)\left[-\frac{1}{2m\cdot 2!} - \frac{B_1}{m^2\cdot 2!}\right] + x^4f^{(\text{iv})}(a) \\
&\times\left[-\frac{1}{2m\cdot 3!} - \frac{B_1}{m^2\cdot 2!\cdot 2!}\right] + x^5f^{(\text{v})}(a)\left[-\frac{1}{2m\cdot 4!} - \frac{B_1}{m^2\cdot 2!\cdot 3!} + \frac{B_2}{m^4\cdot 4!}\right] \\
&+ x^{2n+2}\Bigg[-\frac{1}{2m\cdot(2n+1)!}f^{(2n+2)}(a+\theta_1x) - \frac{B_1}{m^2\cdot 2!(2n)!}f^{(2n+2)}(a+\theta_2x) \\
&+ \frac{B_2}{m^4\cdot 4!(2n-2)!}f^{(2n+2)}(a+\theta_4x) - \cdots + (-1)^n\frac{B_n}{m^{2n}\cdot(2n)!\cdot 2!}f^{(2n+2)}(a+\theta_{2n}x)\Bigg].
\end{aligned}
$$

If now we add to the sum of the first members of (3) and (5) the terms $f(a)$ and R_{2n+2} we reproduce the second member of (1). Hence by (1),

$$
f(a+x) = f(a) + \text{second member of (3)} + \text{second member of (5)} + R_{2n+2}.
$$

In order to simplify the sum of these second members, let us consider the coefficients of the two expressions $x^{2r}f^{(2r)}(a)$ and $x^{2r+1}f^{(2r+1)}(a)$. In the sum of the second members of (3) and (5) the coefficient of $x^{2r}f^{(2r)}(a)$ is

$$
(6)\quad \begin{aligned}
&\frac{1^{2r-1} + 2^{2r-1} + \cdots + m^{2r-1}}{m^{2r}\cdot(2r-1)!} - \frac{1}{2m\cdot(2r-1)!} - \frac{B_1}{m^2\cdot 2!(2r-2)!} \\
&\qquad + \frac{B_2}{m^4\cdot 4!(2r-4)!} - \cdots + (-1)^{r-1}\frac{B_{r-1}}{m^{2r-2}\cdot 2!(2r-2)!}.
\end{aligned}
$$

The coefficient of $x^{2r+1}f^{(2r+1)}(a)$ is

$$\frac{1^{2r}+2^{2r}+\cdots+m^{2r}}{m^{2r+1}\cdot(2r)!}-\frac{1}{2m\cdot(2r)!}-\frac{B_1}{m^2\cdot 2!(2r-1)!}+\frac{B_2}{m^4\cdot 4!(2r-3)!}$$
$$(7)\qquad -\cdots+(-1)^{r-1}\frac{B_{r-1}}{m^{2r-2}\cdot(2r-2)!2!}+(-1)^r\frac{B_r}{m^{2r}\cdot(2r)!}.$$

But by a theorem due to Bernouilli we have

$$1^r+2^r+3^r+\cdots+m^r=\frac{m^{r+1}}{r+1}+\frac{1}{2}m^r+\frac{r!}{(r-1)!2!}B_1m^{r-1}$$
$$(8)\qquad -\frac{r!}{(r-3)!4!}B_2m^{r-3}+\frac{r!}{(r-5)!6!}B_3m^{r-5}\cdots,$$

the second member ending with

$$(-1)^{\frac{r-2}{2}}\frac{r!}{1!\,r!}B_{\frac{r}{2}}m,$$

if r is even and with

$$(-1)^{\frac{r-3}{2}}\frac{r!}{2!\,(r-1)!}B_{\frac{r-1}{2}}m^2,$$

if r is odd. (See Chrystal's Algebra, vol. II, page 209 (2).)

Replacing r by $2r-1$ and substituting in (6) we have for the coefficient of $x^{2r}f^{(2r)}(a)$ an expression which readily reduces to $1/(2r)!$. If we replace r by $2r$ and substitute in (7) we find that the coefficient of $x^{2r+1}f^{(2r+1)}(a)$ reduces to $1/(2r+1)!$. Hence the relation $f(a+x)=f(a)+$ second member of (3) + second member of (5) + R_{2n+2} will reduce to

$$f(a+x)=f(a)+xf'(a)+\frac{x^2}{2!}f''(a)+\cdots+\frac{x^{2n+1}}{(2n+1)!}f^{(2n+1)}(a)$$
$$+\frac{x^{2n+2}}{m^{2n+2}\cdot(2n+2)!}[f^{(2n+2)}(a+\varphi_1x)+2^{2n+1}f^{(2n+2)}(a+\varphi_2x)+\cdots$$
$$(9)\qquad +m^{2n+1}f^{(2n+2)}(a+\varphi_mx)]+x^{2n+2}\Big[-\frac{1}{2^m\cdot(2n+1)!}f^{(2n+2)}(a+\theta_1x)$$
$$-\frac{B_1}{m^2\cdot 2!\,(2n)!}f^{(2n+2)}(a+\theta_2x)+\frac{B_2}{m^4\cdot 4!\,(2n-2)!}f^{(2n+2)}(a+\theta_4x)$$
$$-\cdots+(-1)^n\frac{B_n}{m^{2n}\cdot(2n)!\,2!}f^{(2n+2)}(a+\theta_{2n}x)\Big]+R_{2n+2}.$$

Equation (9) is identical with Taylor's development up to the term containing x^{2n+1}; therefore the remainder after this term in (9) may be equated to the well-known remainder

$$\frac{x^{2n+2}}{(2n+2)!}f^{(2n+2)}(a+\theta x),\qquad 0<\theta<1,$$

after the corresponding term in Taylor's series. This readily yields the value of R_{2n+2} which is sought:

$$
\begin{aligned}
R_{2n+2} = {} & \frac{x^{2n+2}}{(2n+2)!} f^{(2n+2)}(a+\theta x) - \frac{x^{2n+2}}{m^{2n+2}\cdot(2n+2)!}[f^{(2n+2)}(a+\varphi_1 x) \\
& + 2^{2n+1} f^{(2n+2)}(a+\varphi_2 x) + \cdots + m^{2n+1} f^{(2n+2)}(a+\varphi_m x)] \\
& + x^{2n+2}\left[\frac{1}{2m\cdot(2n+1)!} f^{(2n+2)}(a+\theta_1 x) + \frac{B_1}{m^2\cdot 2!\,(2n)!} f^{(2n+2)}(a+\theta_2 x)\right. \\
& - \frac{B_2}{m^4\cdot 4!\,(2n-2)!} f^{(2n+2)}(a+\theta_4 x) + \cdots \\
& \left. - (-1)^n \frac{B_n}{m^{2n}\cdot(2n)!\,2!} f^{(2n+2)}(a+\theta_{2n} x)\right].
\end{aligned}
\tag{10}
$$

In order to simplify this result we shall make use of the following lemma:

If $a_1, a_2, \cdots, a_i$ are all positive and $\varphi_1, \varphi_2, \cdots, \varphi_i$ are all positive and less than unity, then there exists a positive φ, also less than unity, such that

$$
\begin{aligned}
& a_1 F(a+\varphi_1 x) + a_2 F(a+\varphi_2 x) + \cdots + a_i F(a+\varphi_i x) \\
& \qquad = (a_1 + a_2 + \cdots + a_i) F(a+\varphi x),
\end{aligned}
\tag{11}
$$

provided that the function F is continuous.

This may be proved thus: If A and B are the greatest and least values of F in the first member of (11), then is this member increased by replacing F by A and decreased by replacing F by B. Hence, some value, say $F(a+\varphi x)$, between these limits may replace each F in (11) without altering the value of the expression. Hence the theorem. Evidently the proposition also holds if all the a's are negative.

Applying this lemma to equation (10), we may replace each $f^{(2n+2)}$ in the first brackets by $f^{(2n+2)}(a+\varphi x)$. The coefficient of $x^{2n+2} f^{(2n+2)}(a+\varphi x)$ then is

$$
-\frac{1}{m^{2n+2}\cdot(2n+2)!}(1 + 2^{2n+1} + 3^{2n+1} + \cdots + m^{2n+1}).
$$

Changing the form of this coefficient by (8) and substituting in (10) we have

$$
\begin{aligned}
R_{2n+2} = {} & x^{2n+2}\left[\frac{1}{(2n+2)!} f^{(2n+2)}(a+\theta x) - \frac{1}{(2n+2)!} f^{(2n+2)}(a+\varphi x)\right. \\
& - \frac{1}{2m\cdot(2n+1)!} f^{(2n+2)}(a+\varphi x) - \frac{B_1}{m^2\cdot 2!(2n)!} f^{(2n+2)}(a+\varphi x) + \cdots \\
& + (-1)^n \frac{B_n}{m^{2n}\cdot(2n)!\,2!} f^{(2n+2)}(a+\varphi x) + \frac{1}{2m\cdot(2n+1)!} f^{(2n+2)}(a+\theta_1 x)
\end{aligned}
\tag{12}
$$

$$+\frac{B_1}{m^2\cdot 2!(2n)!}f^{(2n+2)}(a+\theta_2 x)-\frac{B_2}{m^4\cdot 4!(2n-2)!}f^{(2n+2)}(a+\theta_4 x)+\cdots$$

$$\left.-(-1)^n\frac{B_n}{m^{2n}\cdot(2n)!\,2!}f^{(2n+2)}(a+\theta_{2n}x)\right].$$

It will be noticed that this remainder may have its terms grouped into two sets whose terms correspond, term of one to term of the other, and such that all the terms of one set have positive, and all those of the other have negative, coefficients. Then by the lemma already proved, the f's in the two sets may be replaced respectively by $f^{(2n+2)}(a+\varphi_1 x)$ and $f^{(2n+2)}(a+\varphi_2 x)$ where φ_1 and φ_2 are positive and less than unity. Making these changes in (12) we have the following final value for the remainder term:

$$R_{2n+2}=x^{2n+2}\left[\frac{1}{(2n+2)!}+\frac{1}{2m\cdot(2n+1)!}+\frac{B_1}{m^2\cdot 2!(2n)!}+\frac{B_2}{m^4\cdot 4!(2n-2)!}\right.$$

$$(13)\qquad \left.+\cdots+\frac{B_{n-1}}{m^{2n-2}\cdot(2n-2)!\,4!}+\frac{B_n}{m^{2n}\cdot(2n)!\,2!}\right]$$

$$\times\,[f^{(2n+2)}(a+\varphi_1 x)-f^{(2n+2)}(a+\varphi_2 x)].$$

NEW BOOKS.

Under this heading will be listed all new books received, which pertain to the general subject of mathematics as related to the collegiate and advanced secondary fields. So far as possible either short descriptive notices or more extended reviews will be given, according to the judgment of the committee having this department in charge. The chairman of this committee is Professor W. H. Bussey, of the University of Minnesota, Minneapolis, Minn., and the other members are Professor C. H. Ashton, University of Kansas, Professor W. C. Brenke, University of Nebraska, and Professor L. C. Karpinski, University of Michigan.

Publishers desiring to use this means of conveying to teachers of mathematics information concerning new books will please forward such books to the chairman of the committee.

TWO NEW BOOKS ON THE CALCULUS.

By W. H. BUSSEY, University of Minnesota.

Differential and Integral Calculus. An introductory course for colleges and engineering schools. By LORRAIN S. HULBURT. Longmans, Green, and Co., New York, 1912. xviii+481 pages.

The Calculus. By ELLERY WILLIAMS DAVIS, assisted by WILLIAM CHARLES BRENKE. Edited by EARLE RAYMOND HEDRICK. The Macmillan Company, New York, 1912. xx+384+63 pages.

In many American colleges and universities, differential and integral calculus are still taught as separate subjects. It is the traditional method. The more modern tendency is to teach a course in "The Calculus," the two branches of the subject being more or less closely correlated according to the text book used. Whether a teacher gives a course of the one kind or the other depends sometimes upon his individual preference, and sometimes upon peculiar conditions in the college or in the department of mathematics. As a usual thing, the same book is not well adapted to the two kinds of courses. The two books which are the subject of this review represent the two methods of teaching the subject.

Hulburt's Calculus keeps differential and integral calculus quite separate. The Davis Calculus correlates the two branches of the subject. It is the opinion of the reviewer that each is an excellent book of its kind. The teacher who is looking for a new text book has a choice between two widely different types. They differ in point of view as well as in method. The fundamental idea of the Davis Calculus is that the derivative is the rate of change of one quantity with respect to another. The whole subject is evolved from the idea of a rate as a limit. The very first paragraph of the book defines the calculus as the study of the relations between interdependent quantities with special reference to their rates of change. The derivative as the slope of a curve and the derivative as speed are regarded as special cases of the derivative as rate of change. In the Hulburt Calculus, on the other hand, the interpretation of the derivative as a rate appears for the first time on page 103, in the chapter on "The derivative as velocity." For the first eight chapters, the derivative is interpreted as the slope of the curve represented by the function. The tendency of the Davis Calculus is to correlate the various parts of the subject, while that of Hulburt's Calculus is to keep them separate.

Another fundamental difference between the books is illustrated by the two ways in which the authors introduce the indefinite integral. In the Hulburt book, the indefinite integral of a function is an abstract thing. The problem is that of finding a function whose derivative is given. The student learns the technique of abstract integration, and then he studies its applications to geometry and kinematics. The advantage of this method is that difficulties of different kinds are met and overcome separately. In the Davis book, on the other hand, the indefinite integral is given a concrete interpretation when it is introduced. The problem is a real problem, namely that of finding the amount of a variable

quantity whose rate of change is known. The work is vitalized by having a meaning attached to it from the beginning. That is an advantage. The student sees what it is for. The disadvantage is that he has the difficulties of interpretation and the difficulties of technique to meet at the same time. This is a typical example of an important difference between the two books. Each type of book has its own peculiar advantages and disadvantages.

Hulburt's Calculus.

Hulburt's Calculus may be thought of as divided into four parts. Book I (174 pages) deals entirely with differential calculus, Books II and III (118 pages) deal entirely with integral calculus. Books IV and V (127 pages) discuss the following topics in both differential and integral calculus, which were omitted in the earlier part of the book: partial and total derivatives; multiple integrals; centers of mass; the theorems of Taylor and Maclaurin; De Moivre's theorem and hyperbolic functions; integration of rational fractions; and envelopes. There is some correlation of the two branches of the calculus in the working out of these topics, but not much. Integration is used in the study of the hyperbolic functions, though it is needed for some of the examples given in connection with envelopes. Book VI (31 pages) is an introduction to ordinary differential equations.

Book I has an introduction dealing with functional symbols, kinds of functions, and graphs of functions with special reference to their discontinuities. Chapter I deals more extensively than is usual with limits and elementary means of finding limits. A strong feature of the book is the way in which it connects with analytic geometry. The derivative is defined in Chapter III. It is immediately interpreted as the slope of the curve represented by the function, and, except for the practical problems in maxima and minima, the applications given in the first eight chapters are all to analytic geometry, especially to curve tracing. Especially worthy of note is the paragraph in Chapter I which explains the distinction between the limit of $f(x)$ when $x \doteq a$ and the value of $f(x)$ when $x = a$. In Chapter VI, the author devotes four pages to the graphical method of finding approximately the roots of algebraic and transcendental equations.

Logarithmic and exponential functions are differentiated for the first time in Chapter VIII. A few pages are given to the interesting compound interest law. Differentials are introduced in a satisfactory way in this chapter. Up to this point, the author has used the D_xy notation for derivative. His remarks in introducing the new notation are interesting: "It is evident that differentials might have been introduced much earlier in the course—in fact, at any stage after the definition of the derivative. The reader must not infer that they are introduced here because any peculiar advantage is to accrue from their use in the following pages. The differential notation possesses few advantages over the notation that we have employed up to this point, and throws no new light upon any of the problems of mathematics. Indeed it has difficulties of its own and can produce confusion of mind in the beginner. But it is a part of our mathe-

matical inheritance from the early days of the calculus, and is still widely used, especially by writers on applied mathematics. Because, then, of their use in modern text books, and for this reason only, the student must become familiar with differentials. In the following pages, therefore, we shall frequently give the differential formulæ along with the derivative formulæ."

Up to this point, the derivative has been interpreted only as the slope of a curve. Chapter IX is entitled "The derivative as velocity." Applications to simple harmonic motion are given.

These first nine chapters constitute a good short course (114 pages). The student will know what differential calculus is good for, even if he never reads farther. The next seven chapters are short, They deal with applications to geometry and kinematics. The last chapter in Book I takes up the law of the mean and indeterminate forms. Partial and total derivatives occur in Book IV. They can be taken up as soon as Book I has been finished, if it is so desired. They are placed in Book IV because an introduction to solid analytic geometry is given in Book III. The theorems of Taylor and Maclaurin and their applications are also in Book IV. They can be taken up before integral calculus is begun if it is desired.

Integral calculus begins in Book II. The first principles of integration, including the fundamental formulæ and the simple devices for bringing the integrand into a form to which one of them will apply, are given in 15 pages. The author then explains how a table of integrals is used, and he suggests that the student buy one. There is no such table in the book. The table constantly referred to in the text is that of B. O. Peirce. The author believes that the student should use such a table freely, so as to have more time to spend on the applications of integral calculus. Nevertheless, he realizes that the student must know some of the devices for finding integrals if he is to be able to use the table intelligently. So he puts in eight pages on special methods of integration, including integration by parts and by algebraic and trigonometric substitution.

All this work in integral calculus takes just 26 pages. The next two chapters (23 pages) are devoted to applications of indefinite integrals to plane geometry and kinematics. Then comes Chapter XXII, dealing with definite integrals and their use in finding lengths of curves and areas under curves. It is in Chapter XXIII that the definite integral is thought of primarily as the limit of a sum. The author gives a geometric proof of the fundamental theorem that the sum $\sum_a^b f(x)\Delta x$ has a definite limit when $f(x)$ is real, single valued, and continuous throughout the interval $a \leqq x \leqq b$.

Book II closes with applications of definite integrals to solids and surfaces of revolution. Book III contains an introduction to solid analytic geometry, and applications of definite integrals to volumes and areas. Only single integration is used. Multiple integrals and their applications are found in Book IV, after partial derivatives. The miscellaneous topics in Book V have already been mentioned. Book VI is an introduction to ordinary differential equations.

Answers to all problems are given at the end of the book.

The Davis Calculus.

The Davis Calculus is the first of a new series of mathematical books to be published by The Macmillan Company under the general editorship of Professor E. R. Hedrick, of the University of Missouri.

The first paragraph of the preface sounds the key-note of the book. "The significance of the calculus, the possibility of applying it in other fields, its usefulness, ought to be kept constantly and vividly before the student during his study of the subject, rather than to be deferred to an uncertain future." The subject is made concrete from the beginning. The derivative is defined as the limit of the quotient of two infinitesimal differences Δy and Δx, but it always means the instantaneous rate of change of the variable y with respect to x. The meaning is made clear before the formal definition is given. Applications to geometry, kinematics, maxima and minima, etc., take up most of the first 90 pages. Not many of these pages are given to the technique of differentiation, because only algebraic functions are used. The derivatives of logarithmic, exponential, and trigonometric functions are taken up for the first time on page 130.

The differential notation is used from the beginning of the book. Differentials are defined in a satisfactory way in the third chapter. In leading up to this subject, the author remarks that, although the derivative is the limit of a fraction rather than a fraction, it may be thought of as a fraction by providing it with a convenient denominator, just as any other quantity may be thought of as a fraction if provided with a convenient denominator. The way in which the author makes this clear to the student is very ingenious. It certainly makes differentials concrete things.

In Chapter V, the indefinite integral is introduced in connection with the problem of the reversal of rates. The idea of rate is very much in evidence in the first applications of integrals to the finding of lengths of curves and areas under curves. In fact the fundamental summation formula which says that a definite integral is the limit of a sum is obtained from the interpretation of an indefinite integral as a reversed rate. A little later, the author remarks that "the greater number of integrations appear more naturally as limits of sums than as reversed rates," and he adds this foot note: "It is really a waste of time to discuss at great length which fact about integrals is used as a definition, and which one is proved; to satisfy the demand for formal definition, the integral may be defined in either way,—as a limit of a sum, or as a reversed differentiation. The important fact is that the two ideas coincide, which is the fact stated in the summation formula." The work on the definite integral as the limit of a sum is preceded by a few pages devoted to elementary methods of approximate summation as distinguished from the exact summation given by definite integrals. Applications of Cavalieri's theorem and the prismoid formula are included in this chapter.

One of the good things about this book is that the first five chapters (130

pages) contain so many useful and interesting applications of both differential and integral calculus with a minimum of the technique of differentiation and integration.

Beginning with Chapter VI, differential and integral calculus are not kept separate. The derivatives of log x and sin x which occur for the first time in this chapter, are not obtained in the usual way. The derivative of $\log_{10} x$ is found before $\log_e x$ by a method which lays stress on the very meaning of logarithms and is as rigorous as the traditional proof. The derivative of sin x is found by using the properties of circular motion. The logarithmic derivative, which is usually a mere device for finding the actual derivative, is interpreted as the relative rate of change of a quantity y with respect to a quantity x. This is another example of the author's desire to have nothing abstract. Another instance of the same tendency is the definition of curvature as the instantaneous rate of change of α per unit length of arc, where α is the angle made by the tangent with the x-axis or the polar axis.

Chapter VII deals with the technique of integration. There is a good table of integrals in the back of the book, and the student is expected to use it intelligently, not slavishly. This chapter contains applications of single and multiple integrals, and a brief treatment of improper integrals with precautions to be observed in using them.

The first seven chapters (226 pages) have been developed without the law of the mean and the theorems of Taylor and Maclaurin. The manner in which these are introduced and used is one of the strong features of the Davis Calculus. Chapter VIII is on methods of approximation. It begins with a discussion of the elementary methods of finding equations to represent, at least approximately, curves which are given empirically. Lagrange's interpolation formula, the method of differences, and logarithmic plotting are used. Closely related to this subject is the problem of finding polynomial approximations of functions which are not polynomials. Taylor's Theorem is introduced for this purpose. Taylor's series with the remainder is taken up before there is any mention of infinite series, and a study is made of the error involved in neglecting the remainder and using the resulting polynomial approximation for the function under consideration. The work on infinite series is not extensive. Convergence and divergence are defined and illustrated, and a general test for the convergence of the series furnished by the theorems of Taylor and Maclaurin is developed. There is a good paragraph on precautions to be taken in using infinite series.

Chapter IX deals with partial and total derivatives, with applications to plane and solid geometry and to physics. Chapter X is an introduction to ordinary differential equations of the first order. The book closes with 58 pages of useful tables which will relieve the student of many difficulties if he uses them intelligently.

An answer book, containing answers to all of the numerical problems in the book, and indicating in some detail the methods of attack for the theoretical problems, will be published some time this winter. The "Errata," which will

also be printed on a separate sheet, will be printed as a page of the answer book.

At the end of this review of two new books on the calculus, the two lines from Pope, quoted at the beginning of the Davis Calculus, seem appropriate.

> "Be not the first by whom the new is tried,
> Nor yet the last to lay the old aside."

PROBLEMS AND QUESTIONS.

Under this heading the MONTHLY will act as a medium for the interchange of ideas on behalf of its readers. While questions proposed merely as puzzles will not necessarily be barred by the editors, yet it is hoped that light on many real difficulties may be sought through these columns; and that many problems arising in the actual processes of investigation may be proposed for solution. In order to encourage this latter form of questions and problems, the names of proposers will not be given when the editors are requested to withold them.

A committee of the Editorial Board, consisting of Professor B. F. Finkel, chairman, Drury College, Springfield, Mo.; Professor R. P. Baker, University of Iowa, and Professor R. D. Carmichael, Indiana University, will have charge of contributions to this department. Correspondence may be addressed to the chairman or to any member of the committee. No problems are printed this month owing to the unusual space occupied by other matter. This department will be resumed in the February issue.

NEWS AND NOTES.

Under this heading the MONTHLY will print information concerning new appointments and promotions of teachers of mathematics, notices of meetings of mathematical associations, biographical and bibliographical notes, references to articles of interest in other mathematical journals, courses in mathematics offered in American universities and any items whatsoever of general interest to teachers of mathematics.

Contributions to this column should be sent to Professor FLORIAN CAJORI, Colorado College, Colorado Springs, Colo., who is chairman of the committee having this department in charge. Other members are Professor G. A. MILLER, University of Illinois, Professor W. H. BUSSEY, University of Minnesota, and Professor W. DE W. CAIRNS, Oberlin College.

Mr. E. R. SMITH, formerly of the Brooklyn Polytechnic Institute, is now head master of the Park School, Baltimore, Md.

Mr. GUY H. ALBRIGHT, assistant professor of mathematics at Colorado College will spend the second semester of the present academic year at Harvard University and will give his time mainly to the pursuit of graduate work.

Professor G. A. MILLER, of the University of Illinois, has been elected corresponding member of the Spanish Mathematical Society.

Professor DAVID EUGENE SMITH, of Columbia University, delivered an address on "The Teaching of Mathematics" at the annual meeting of the Kansas State Teachers' Association held at Topeka in November, 1912.

Professor H. E. SLAUGHT, of the University of Chicago, delivered an address on "The Present Trend in the Teaching of Secondary Mathematics" at the Iowa State Teachers' Association held at Des Moines in November, 1912.

At the meeting of the college section of the Illinois State Teachers' Association, held at Peoria December 27, 1912, "The Preservation of the Four Years' College Course" was a topic for discussion led by President THOMAS McCLELLAND of Knox College.

The winter meeting of the Chicago Section of the American Mathematical Society was merged in the annual meeting of the Society, held at Cleveland, Ohio, December 31, 1912, to January 2, 1913. An account of this meeting will be given in the February issue of the MONTHLY.

Mr. WILLIAM BETZ, head of the department of mathematics in the East High School, Rochester, N. Y., addressed the meeting of the Principals' Association of New York State at Rochester, December 27, 1912, on the "Final Report of the National Committee of Fifteen on Geometry Syllabus."

The *Mathematics Teacher* for December, 1912, contained the report of the National Committee of Fifteen on Geometry Syllabus in its final form, as presented and accepted at the Chicago meeting of the National Education Association in July, 1912. A large edition of reprints of this report is now ready for distribution through the Commissioner of Education, Washington, D. C. These may be had *gratis* upon application.

The Southern Association of Colleges and Preparatory Schools held its annual meeting at Spartanburg, S. C., in November, 1912. Membership in this association is based upon the standards set by the Carnegie Foundation for the advancement of teaching as applied to both colleges and secondary schools. The only college added to the membership at this meeting was Converse College for women at Spartanburg, where Professor T. McNIDER SIMPSON is the aggressive head of the department of mathematics.

The following letter from Professor B. F. YANNEY, of the University of Wooster, Ohio, speaks for itself: "I note with interest and pleasure the changes proposed in respect to the AMERICAN MATHEMATICAL MONTHLY. The new control makes secure its future career. The wider scope of its aims insures still greater usefulness to its patrons. It has been an important source of help to me during the entire period of its existence thus far, and now it promises to be even more so in the years to come. It deserves a steadily growing success, which is my wish and prediction for it."

Mr. MIKAMI shows in the *Archiv der Mathematik und Physik,* volume XX, page 1, that the proof of the Pythagorean theorem usually attributed to the Hindu Bhaskara (about 1150 A. D.)—based on a figure in which a right triangle is drawn four times in the square on the hypotenuse, so that in the middle there remains a square whose side equals the difference between the two legs of the right triangle—was given early in the Christian era by the Chinese Chang Chun-Ch'ing.

Under the general title "Present Civilization" (Kultur der Gegenwart), the well-known German publisher, B. G. TEUBNER of Leipzig has announced a long series consisting of sixty volumes. Eighteen of these are to be devoted to the mathematical and the natural sciences. The part devoted to the mathematical sciences is already in press. It is edited by F. Klein, with the collaboration of P. Stäckel, H. E. Timerding, A. Voss, and G. H. Zeuthen.

A letter from B. G. Teubner dated November 23, 1912, states that the remaining two parts of Pascal's "Repertorium der höheren Mathematik" can not appear within the next two or three months. Recent circulars issued by the publishers stated that this work would appear at the end of 1912. The new edition consists of four volumes, and gives in a convenient form a statement of the fundamental theorems of pure mathematics. A similar work devoted to applied mathematics, entitled "Repertorium der Angewandte Mathematik" is also to appear in four volumes, the first of which is expected in the spring of 1913.

The October number of the journal of the new Spanish Mathematical Society, entitled "*Revista de la Sociedad Matematica Espanola,*" calls attention to the fact that there were twenty-five Spanish mathematicians at the International Mathematical Congress held at Cambridge, England, last August. The numbers of the Spanish mathematicians attending the four preceding international mathematical congresses were 1, 3, 1, and 5 respectively. It would appear that the mathematical interest in Spain is rapidly growing at the present time, and it is to be hoped that this will continue to grow under the influence of the new national mathematical society.

A new weekly scientific journal called the *Natur Wissenschaften* is to be started in Germany in January, 1913. This journal is expected to occupy the field covered by *Nature* in England, and *Science* in America. The technical applications of the sciences are, however, to be considered more fully than in these English and American journals, and mathematics is to receive as much attention as may be possible in such a general journal.

The annual meeting of the Southwestern Section of the American Mathematical Society was held at Lincoln, Neb., November 30, 1912. Papers were presented by Professor W. H. Roever of Washington University, Professor S. W. Reaves of the University of Oklahoma, Dr. E. R. Bennett of the University of Nebraska (two papers), Professor Arnold Emch of the University of Illinois, Professor R. D. Carmichael of Indiana University, Professor Florian Cajori of Colorado College, Dr. S. Lefschetz of the University of Nebraska, Dr. E. L. Dodd

of the University of Texas, Dr. T. H. Gronwall, Chicago, Ill. (four papers), Professor E. R. Hedrick of the University of Missouri, and Professor Louis Ingold of the University of Missouri. Abstracts of these papers will be printed in the *Bulletin* of the American Mathematical Society.

Of the 177 publications of the Carnegie Institution at Washington, as shown in the latest catalog, five are in pure mathematics as follows: "Synopsis of Linear Associative Algebra" by James Byrnie Shaw, University of Illinois; "Factor Table for the First Ten Millions" and also "Tables Giving a Complete List of Prime Numbers between the Limits 1 and 10,006,721," by Derrick N. Lehmer, University of California; "A Sylow Factor Table of the First Twelve Thousand Numbers" by Henry W. Stager, University of California; and "The Symmetric Function Tables of the Fifteenthic" by Floyd F. Decker, Syracuse University. There are also some other papers of a mathematical nature including one by F. R. Moulton of the University of Chicago on "Periodic Orbits" and several in collaboration with Professor T. C. Chamberlin on "Contributions to Cosmogony and the Fundamental Problems of Geology." The latter are by William D. MacMillan, Arthur C. Lunn, and F. R. Moulton of the University of Chicago, and Charles S. Slichter of the University of Wisconsin.

An outline of the report presented by A. Gutzmer at the Mathematical Congress in Cambridge, England, on the work done by the German sub-committee of the International Commission on the Teaching of Mathematics, is printed in *Science* for December 13, 1912. The German report is composed of five volumes: (1) The secondary schools of northern Germany; (2) The secondary schools of southern and middle Germany. (3) Special problems of secondary mathematics instruction; (4) Mathematics in the technical schools; (5) The teaching of mathematics in elementary schools, and in the training schools for elementary teachers. A significant paragraph in this German report is the following: "This reorganization of secondary education aims at making the youth of our country sympathetic with labor as well as appreciative of the best in modern culture. From this point of view the teaching of mathematics and science assumes a position equivalent to that in history and languages."

It is just announced, as we go to press, that Doctor Oskar Bolza, Honorary Professor of Mathematics, University of Freiburg, Freiburg, Germany, will lecture at the University of Chicago during the summer session of 1913. The topics upon which he will lecture are: Linear Integral Equations and Theory of Functions of a Complex Variable. This is his first return since his retirement from Chicago as Non-Resident Professor. His many friends in this country will welcome him most heartily.

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HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

By FLORIAN CAJORI, Colorado College.

THE MODERN EXPONENTIAL NOTATION
(Continued).

J. H. Rahn's *Deutsche Algebra*, printed in Zürich, 1659, contains for positive integral powers two notations, one using the cartesian exponents, a^3, x^4, the other consisting in writing a small spiral between the base and the exponent on the right. Thus a ↄ 3 signifies a^3. The spiral signifies *involution*, a process which he calls *involviren*. An English translation, altered and augmented by John Pell, was made by Thomas Brancker and published 1668 in London.[1] In the same year positive integral exponents were used by Lord Brouncker in an early volume of the *Philosophical Transactions* of London.[2] In these transactions none of the pre-descartian notations for powers appear, except a few times in an article of 1714, written by John Cotes.

Of interest is the following passage in Newton's *Universal Arithmetick* (which consists of lectures delivered at Cambridge in the period, 1669–1685 and first printed 1707): "Thus $\sqrt{}\,64$ denotes 8; and $\sqrt{}\,3 : 64$ denotes 4. . . . There are some, that to denote the Square or first Power, make use of q, and of c for the Cube, qq for the Biquadrate, and qc for the Quadrato-Cube, etc. . . . Others make use of other sorts of Notes, but they are now almost out of Fashion."[3] In an edition of 1679 of the works of Fermat the algebraic notation of Vieta, originally followed by Fermat, is discarded in favor of the exponents of Descartes.[4] It would seem, from what has been cited, that about 1660 or 1670 the positive integral exponent had won an undisputed place in algebraic notation. Though generally adopted, it was not universally so. A large volume, the P. Gasparis Schotti *Cursus mathematicus*, Frankfurt a. M., 1661, and the second edition of Diophantus by Bachet de Méziriac (Toulouse, 1670), contain no trace of the modern exponential notation. Joseph Raphson's explanation of his method

[1] See G. Wertheim in *Bibliotheca mathematica*, 3d S., Vol. III, 1902, pp. 113–126.

[2] *Phil. Trans.*, Vol. III, for anno 1668, printed 1669, p. 647.

[3] Newton's *Universal Arithmetick*, London, 1728, p. 7.

[4] *Œuvres de Fermat*. Éd. PAUL TANNERY ET CHARLES HENRY, T. I, Paris, 1891, p. 91 foot-note.

of approximation to the roots of numerical equations, printed in the Latin edition of John Wallis's *Algebra*, in 1693, does not use positive integral exponents; Raphson uses powers of g up to g^{10}, but in every instance he writes out each of the factors, after the manner of Harriot.

It is worthy of note that for a long while there were two different notations for the *square* of a letter. Some wrote aa; others a^2. It would be rather difficult to make out a clear case in favor of a^2, were one to base the argument on the greater economy of space. The symbolism aa was preferred by Descartes, Huygens, Rahn, Kersey, Wallis, Newton, Halley, Rolle, L. Euler — in fact, by most writers of the second half of the seventeenth and of the eighteenth centuries; a^2 was preferred by Leibniz, Ozanam, David Gregory.

Negative and fractional exponential notations had been suggested by Chuquet, Stevin and others. The modern symbolism is due to Wallis and Newton.

In his *Arithmetica infinitorum*, Oxford, 1656, Wallis uses positive integral exponents and speaks of negative and fractional "indices."[1] But he does not actually write a^{-1} for $1/a$, or $a^{\frac{3}{2}}$ for $\sqrt{a^3}$. He speaks of the series $1/\sqrt{1}$, $1/\sqrt{2}$, $1/\sqrt{3}$, etc., as having the "index $-\frac{1}{2}$," the series 1, 4, 9, $\cdots$ as having the "index 2," the series $\sqrt{1}$, $\sqrt{8}$, $\sqrt{27}$, $\cdots$ as having the "index $\frac{3}{2}$."[2] Our modern notation involving fractional and negative exponents was formally proposed about a dozen years later. On June 13, 1676, Newton wrote to H. Oldenburg, then secretary of the Royal Society of London, a letter which was forwarded to Leibniz. The letter contains the following passage, which is interesting as containing the binomial theorem and explaining the use of negative and fractional exponents:

Sed extractiones radicum multum abbreviantur per hoc Theorema.

$$\overline{P + PQ}|^{m/n} = P^{m/n} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.$$

Ubi $P + PQ$ significat quantitatem, cujus radix, vel etiam dimensio quævis, vel radix dimensionis, investiganda est; P, primum terminum quantitatis ejus; Q, reliquos terminos divisos per primum. Et m/n, numeralem indicem dimensionis ipsius $P + PQ$: sive dimensio illa integra sit; sive, ut ita loquar, fracta; sive affirmativa, sive negativa. Nam, sicut analystæ, pro aa, aaa, &c. scribere solent a^2, a^3, &c. sic ego, pro $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{ca^5}$, &c. scribo $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$; & pro $1/a$, $1/aa$, $1/aaa$, scribo a^{-1}, a^{-2}, a^{-3}.[3]

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[1] J. Wallis, *Arithmetica infinitorum*, 1656, p. 80, Prop. CVI.

[2] Of interest is the following quotation from a discussion by T. P. Nunn, in the *Mathematical Gazette*, Vol. VI, 1912, p. 255: "Those who are acquainted with the work of John Wallis will remember that he invented negative and fractional indices in the course of an investigation into methods of evaluating areas, etc. He had discovered that if the ordinates of a curve follow the law $y = kx^n$, its area follows the law $A = 1/(n+1) \cdot kx^{n+1}$, n being (necessarily) a positive integer. This law is so remarkably simple and so powerful as a method that Wallis was prompted to inquire whether cases in which the ordinates follow such laws as $y = k/x^n$, $y = k\sqrt[n]{x}$ could not be brought within its scope. He found that this extension of the law would be possible if k/x^n could be written kx^{-n}, and $k\sqrt[n]{x}$ as $kx^{1/n}$. From this, from numerous other historical instances and from general psychological observation, I draw the conclusion that extensions of notation should be taught because and when they are needed for the attainment of some practical purpose, and that logical criticism should come *after* the suggestion of an extension to assure us of its validity."

[3] Isaaci Newtoni *Opera* (ed. S. Horsley), Tom. IV, Londini, 1782, p. 525.

It is worthy of note that the modern fractional exponent was first introduced by Newton in the announcement of his Binomial Theorem, invented by him some time before 1669. Newton used also negative fractional exponents. In November, 1676, Leibniz collected some of his results on a sheet of paper; he uses here the notation[1] x^{-3}, $x^{-\frac{1}{2}}$. It should be observed also that the use of *literal* exponents is suggested in Newton's form for the binomial theorem as given above, and that literal exponents came to be generally used in the latter part of the seventeenth century by Newton, Leibniz and their followers. Perhaps the earliest occurrence of literal exponents is in Wallis's *Mathesis universalis*, Oxford, 1657, where a few expressions like $\sqrt{^dR^d} = R$, $AR^m \times AR^n = A^2R^{m+n}$ have been noticed.[2]

The theory of exponents, involving positive, negative and fractional values, is explained and freely used in the much respected and widely read work, entitled, *Analyse demontrée*, by Charles Reyneau, Paris, 1708. The theory is explained in the introduction to the first volume. This is done because the treatises on algebra then in use did not usually contain it. Reyneau uses these words: "Le seul calcul qui n'est pas expliqué dans les Traités d'Algebre dont on vient de parler, est celui des exposants des puissances."[3] In deriving rules for differentiation,[4] Reyneau passes from $x^x = a$ to $xlx = la$, and from $x^{x^x} = y^{y^y}$ to $x^xlx = y^yly$.

The interesting question arises, when and where did the union between the exponential and logarithmic concepts take place? It did not occur until the eighteenth century. As is quite proper, there was quite a long courtship. It goes back to the time of Wallis. In the twelfth chapter of his *Algebra*, 1685, Wallis develops the theory of logarithms, beginning with two progressions 1, 2, 4, 8, $\cdots$ and 0, 1, 2, 3, $\cdots$. He then generalizes by taking

$$\begin{array}{l} 1 \cdot r \cdot rr \cdot r^3 \cdot r^4 \cdot r^5 \cdot r^6 \text{ etc.} \\ 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \text{ etc.} \end{array}$$

and remarks that "*these exponents they call logarithms*, which are artificial numbers, so answering to the natural numbers, as that the addition and subduction of these answers to the multiplication and division of the natural numbers." And yet, Wallis does not come out, resolutely, with the modern definition of a logarithm, and use it.

A similar point of view was reached[5] by John Bernoulli I in a letter of May, 1694, addressed to Leibniz. Bernoulli discusses an "ideam novi $\cdots$ *calculi percurrentis*," a terminology later discarded in favor of calculus of "exponential quantities." He speaks of the construction of exponential curves $x^x = y$ by

[1] C. I. Gerhardt, *Der Briefwechsel v. G. W. Leibniz mit Mathematikern*, Bd. I, Berlin, 1899, p. 230.

[2] G. Eneström, *Bibliotheca mathematica*, 3d S., Vol. 9, 1908–1909, p. 329.

[3] Ch. Reyneau, *Analyse demontrée*, Vol. I, Paris, 1708, p. xvii.

[4] *Analyse demontrée*, Vol. II, 1708, p. 806.

[5] Got. Gul. Leibnitii et Johan. Bernoullii *Commercium philosophicum et mathematicum*, T. I, 1745, p. 8.

means of the ordinary logarithmic curve which he says is itself a curve of that type, having the equation $a^x = y$. Bernoulli assumes the logarithmic curve to be drawn and uses the graph for plotting the equation $x^x = y$. He assumes a value x_1, then measures off the ordinate $\log x_1$ on the logarithmic curve and geometrically constructs the product $x_1 \log x_1 = \log y_1$. Finally he finds, again by the aid of the logarithmic curve, the antilogarithm y_1. This y_1, together with the assumed value of x_1, yields a point on the curve $x^x = y$. Thus the curve can be constructed by points. From our point of view, the interest of this process lies in the fact that the logarithmic curve and therefore the logarithm itself, is connected with the equation $a^x = y$. Here x is looked upon as the logarithm of y. Bernoulli makes no mention here of arithmetic and geometric progressions. His procedure involves the modern definition of a logarithm, which, however, he does not explicitly state. The process shows that Bernoulli passed from $x^x = y$ to $x \log x = \log y$, though he did not actually write down this last equation. In June, 1694, Leibniz sent J. Bernoulli a letter in reply, in which he writes[1] both $x^x = y$ and $x \log x = \log y$. We see from the above that Leibniz and J. Bernoulli had a grasp at this time of the exponential *function.*

II. From Leibniz and John Bernoulli I to Euler. 1712–1747.

UNSUCCESSFUL ATTEMPTS TO CREATE A THEORY OF LOGARITHMS OF NEGATIVE NUMBERS.

In the eighteenth century the tendency to take rules derived only for a special case and apply them to more general cases became more pronounced than it had been. It is a tendency which in the nineteenth century came to be called the "principle of the permanence of equivalent forms" or, better still, the "principle of the permanence of formal laws." To-day, we look upon these extensions as things we are at liberty to do or not to do, as we may please. If we find it most convenient in a given research to reject negative and complex numbers and confine ourselves to positive numbers, we may do so, but we are expected to state our position clearly, then to maintain it. In the seventeenth and eighteenth centuries it was not clearly felt that, logically, one had this freedom to extend or to limit the number concept. Negative numbers came to be used freely, yet this extension of the domain was done with misgivings, which show themselves in the names applied to them, such as *false* or *defective* numbers, *numeri ficti.* Still more pronounced was the feeling of discomfort toward bi or $a + bi$; such numbers were called *imaginary, impossible.* It was felt that the validity of negative and complex numbers should be *proved,* not *assumed;* that the rules of operation with such numbers was a matter requiring *demonstration.* Hence the eighteenth century mathematicians, including even men of the type of Laplace, tried to prove the rule of signs in the multiplication of two negative numbers. The "proofs" given were futile; they rested on a syllogism without

[1] Leibnitii et Bernoullii *Com. phil. et math.*, I, p. 10.

a major premise. This difference in the point of view must be borne in mind in the history of the extension of the logarithmic concept to negative and complex numbers. It will help to explain how it was that the controversy on this subject lasted for a whole century and reached well into the nineteenth century.

Several of the eighteenth century mathematicians of the first rank, particularly Euler, used imaginaries freely; some other mathematicians looked upon the imaginaries with suspicion. Here and there an eighteenth century mathematician declaims loudly against the use both of the negative and the imaginary.[1] Interesting is the language used by Leibniz in 1702. He speaks of the imaginary factors of $x^4 + a^4$ as "an elegant and wonderful recourse of divine intellect, an unnatural birth in the realm of thought, almost an amphibium between being and non-being."[2] Most wonderful was the result reached[3] in 1702 by John Bernoulli[4] (1667–1748). He explained the transformation of the differential $adz \div (b^2 + z^2)$ into $-\, adt \div 2bt\sqrt{-1}$, by means of the relation $z = (t - 1) . b\sqrt{-1} \div (t + 1)$ and thereby showed that the integral can be expressed as an arctangent and also as a logarithm. In this manner he pointed out a relation between the logarithm of an imaginary number and the arctangent. This *logarithme imaginaire*, as John Bernoulli called it, was so novel and so foreign to the thought of the time that it caused little comment.

In a letter of the same year (1702), dated June 24 and addressed to John Bernoulli, Leibniz speaks of imaginary logarithms in connection with the problems of integration.[5] There is danger of attaching too much importance to passages of this sort. To Leibniz and J. Bernoulli an imaginary often meant simply non-existence. If logarithms of imaginary numbers were believed to exist, nothing is here brought out as to the nature of such logarithms.

The controversy on logarithms which agitated mathematicians for more than a century did not originate primarily in discussions of imaginary number, but rather in discussions of negative number. Are negative numbers less than nothing? If they are, then in a proportion $1 : -1 = -1 : 1$, the greater number is to the less, as the less is to the greater — an impossibility. This matter was discussed by many writers, including Leibniz, Newton, D'Alembert, Maclaurin, Rolle and Wolf. Leibniz published a paper on this subject in 1712.[6] He considered the above proportion impossible in fact, but maintained that such proportions may be used with the same advantage and safety with which other inconceivable quantities are used. Leibniz said that a ratio may be considered imaginary, when it has no logarithm. The ratio $-1 \div 1$ has no logarithm; for, there would result $\log(-1/1) = \log(-1) - \log 1 = \log(-1)$.

[1] See Cantor, *op. cit.*, Vol. 4, 1908, pp. 79–90.

[2] Leibniz, *Werke*, Ed. Gerhardt, 3. F., Bd. V, 1858, Berlin, p. 357: *Itaque elegans et mirabile effugium reperit in illo Analyseos miraculo, idealis mundi monstro, pene inter Ens et non-Ens Amphibio, quod radicem imaginariam appellamus.*

[3] Joh. Bernoulli, *Opera*, Vol. I, Laus. et Genevæ, 1742, p. 399.

[4] For the sake of distinction, this John Bernoulli is frequently designated as John Bernoulli I.

[5] *Leibnitii et J. Bernoullii Commerc. Phil. et math.*, T. II, 1745, p. 81.

[6] *Acta Eruditorum*, 1712, pp. 167–169; *Werke*, 3. F., Bd. V, Halle, 1858, pp. 387–389.

Leibniz declared that -1 has no real logarithm; such a logarithm could not be positive, for a positive logarithm corresponds to a number larger than 1; the logarithm could not be negative, for a negative logarithm corresponds to a positive number, less than unity. The only alternative remains, therefore, to declare the logarithm of -1 as not really true, but imaginary. He arrives at the same conclusion from the consideration that if there really existed a logarithm of -1, then half of it would be the logarithm of the imaginary number $\sqrt{-1}$, a conclusion which he considered absurd. We notice in these statements of Leibniz a double use of the term imaginary: (1) in the sense of non-existent, (2) in the sense of a number of the type $\sqrt{-1}$.

On March 16 of this year, before the appearance of the article, Leibniz mentioned the subject in a letter to John Bernoulli. That letter opened up a friendly controversy between the two men on the logarithms of negative and imaginary numbers. In their correspondence they debated this question for sixteen months. At that time they were the only ones interested in this question; in fact, they were the only ones to whom the problem of the existence or non-existence of logarithms of negative numbers had occurred. The controversy opens up a number of most interesting points and gives an insight into the algebraic concepts of the time as could not be obtained readily in any other way. For brevity let $+n$ or $+x$ indicate a "positive number," $-n$ or $-x$ a "negative number," $\log(+n)$ the "logarithm of a positive number," $\log(-n)$ the "logarithm of a negative number," i or in, an "imaginary number." The following is a synopsis of the correspondence:[1]

March 16, 1712. Leibniz to J. Bernoulli. This is the letter already referred to. L. says that $-1/1$ is imaginary, since it has no logarithm.

May 25, 1712. J. Bernoulli to Leibniz. B. rejects L.'s proof that the ratio $1 : -1$, or $-1 : 1$ is imaginary, for the reason that $-x$ has a logarithm. We have $dx : x = -dx : -x$; hence, by integration, $\log x = \log(-x)$. The logarithmic curve $y = \log x$ has therefore two branches, symmetrical to the axis Y, just as the hyperbola has two opposite branches.

June 30, 1712. Leibniz to J. Bernoulli: L. repeats his argument that $\log(-2)$ does not exist; for, if it did, its half would equal $\log \sqrt{-2}$, an impossibility. The rule for differentiating, $d \log x = dx : x$, does not apply[2] to $-x$. In the logarithmic curve $y = \log x$, x cannot decrease to 0 and then pass to the opposite side, since the curve cannot cut the Y axis, which is asymptotic to it.

August 13, 1712. J. Bernoulli to Leibniz: The argument that $\log(-2)$ does not exist, because $\log \sqrt{-2}$ does not exist is invalid. I deny that $\log(\sqrt{-2})$ is half of $\log(-2)$, even though it be true that $\log \sqrt{2} = \frac{1}{2} \log 2$. The difference is that $\sqrt{2}$ is a mean proportional between 1 and 2; $\sqrt{-2}$ is not a mean proportional between -1 and -2. Just as $\log \sqrt{1 \times 2} = \frac{1}{2} \log 2$, so is $\log \sqrt{-1 \times -2} = \frac{1}{2} \log(-2)$. That is, $\log \sqrt{2} = \frac{1}{2} \log 2 = \frac{1}{2} \log(-2)$. In passing from $+x$ to $-x$ in a curve it is not necessary that the curve cut the y-axis. Witness the conjugate hyperbola whose abscissas are common to $+$ and $-$ ordinates, but never to vanishing ones. In $y = \log x$, one branch of the curve passes into the other at infinity, when $x = 0$, in the same manner as in the conchoid of Nicomedes and other curves.

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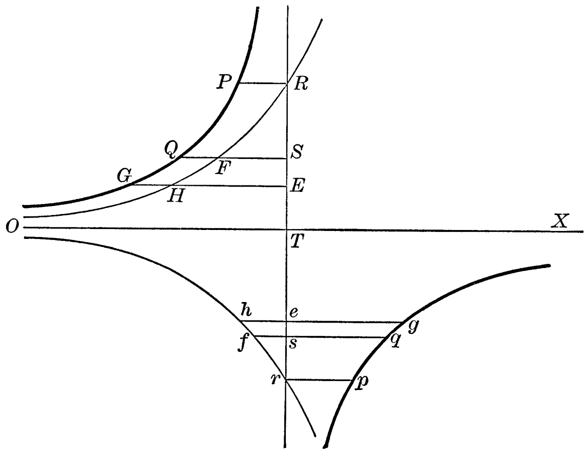
[1] Got. Gul. Leibnitii et Johan. Bernoullii *Commercium Philosophicum et Mathematicum*, Tomus Secundus, Lausannæ et Genevæ, 1745, pp. 269, 276, 278, 282, 287, 292, 296, 298, 303, 305, 312, 315.

[2] *Sed hæc regula, quod differentiale divisum per numerum dat differentiale Logarithmi, et quævis alia de Logarithmorum natura et constructione non habet locum in numeris negativis, ut reperies, ubi demonstrare voles.*

Sept. 18, 1712. Leibniz to J. Bernoulli: Logarithms are numbers in arithmetic progression, corresponding to numbers in geometric progression, of which one number may be 1 and another may be any positive number. Assume $\log 1 = 0$ and $\log 2 = 1$. In the geometric progression thus limited, $-n$ can never be obtained, no matter how many third proportionals are formed. In the series 1, 2, 4, the mean proportional between 1 and 4 is both $+2$ and -2. But -2 cannot be in the same geometric progression which contains $+2$; that is, no value of e makes $-2 = 2^e$ or $e = \log(-2)$; hence there is no logarithm of -2, and a curve $e = \log x$ that is satisfied by $x = 1$ and $x = 2$, cannot be satisfied by $x = -2$. *Otherwise thus:* If -2 has a logarithm, then the half of this logarithm exists and is the logarithm of $\sqrt{-2}$. But $\sqrt{-2}$ is an impossible number; hence, half of $\log(-2)$ is impossible, and the whole, or $\log(-2)$ is impossible. *Another point:* In logarithmic theory, n^e or $\sqrt[e]{n}$ is represented by $\log n \cdot e$, or $\log n : e$; nn or $n : n$ is represented by $\log n \pm \log n$; n is represented by $\log n$; by what is $-n$ represented? There is no mode of representation below the ones already named. *Again:* Granting for the moment that $\log(-2)$ exists, it follows that $\log \sqrt{-2}$ is half of it, for $\sqrt{-2}$ is the mean proportional between $+1$ and -2; hence, $\log \sqrt{-2} = (\log 1 + \log(-2)) : 2 = \frac{1}{2} \log(-2)$.

Nov. 9, 1712. J. Bernoulli to Leibniz: B. says that he sees nothing in the last letter which proves the impossibility of $\log(-n)$. He admits that there is no transition from a (geometric) series of positive terms to one of negative terms, and so $\log(-n)$ does not exist in this case.[1] But negative numbers determine their own peculiar series starting with -1, instead of $+1$ Thereby the same logarithmic properties follow for $-n$ as for $+n$. He reiterates that



$\log n = \log -n$. To show that $y = \log x$ has two branches, he uses the rectangular hyperbola $PQGpqg$ and lets SF and EH be proportional to the hyperbolic areas $RSQP$ and $REGP$. Let PR and GE be constants and FS a variable. As S touches T, FS is infinite and the area is infinite. Now keeping to the same law of generation of the curve RFH, let the point S proceed to e (for what can hinder this?). The area upon Re is partly + and partly −, and equal to EP, when $TE = Te$. We have then $EH = eh$. Similarly, if $Ts = TS$, then $sf = SF$. Thus arises the branch hfr which, with HFR, constitutes the one logarithmic curve, just as the two branches of the hyperbola constitute one curve. If $TR = +1$, $Tr = -1$, $TS = +n$, $Ts = -n$, then $SF = \log n$, $sf = \log(-n)$. As $SF = sf$, we must have $\log n = \log(-n)$.

Jan., 1713. Leibniz to J. Bernoulli: Assuming $2^e = x$, if $x = 1$ then $e = 0$, if $x = 2$ then $e = 1$. When $x = -1$, e cannot be assigned.

[1] *Hoc unum efficis omnibus Tuis argumentis, ut ostendas non dari transitum ex serie numerorum affirmativorum in seriem negativorum, hoc est, assumpta unitate (nempe + 1) pro initio seriei numerorum, nullum numerum negativum ex illa serie inveniri posse, adeoque nullos eorum logarithmos hoc casu existere; quod quidem non nego. Sed hoc non impedit, quominus numeri negativi suam peculiarem constituant seriem, assumta pro eorum initio unitate negativa, (nempe − 1).*

Feb. 28, 1713. J. Bernoulli to Leibniz: If in $v^e = x$ we assume $x = 2$ and $e = 1$, also $x = 1$ and $e = 0$, then truly e cannot be assigned when $x = -1$. As these assumptions are arbitrary, change them so that $x = -1$ when $e = 0$, then e can be assigned for any $-x$.

April 26, 1713. Leibniz to J. Bernoulli: You say that my values for e and x in $2^e = x$ are arbitrary. You let $x = -1$ and $e = 0$. Mine is the most natural. Aside from that consider *firstly* that we cannot have both $\log n$ and $\log(-n)$, for if $\log(-1) = 0$, then $\log(-1)^2 = \log 1 = 2 \times 0 = 0$, and $\log \sqrt{-1} = 0/2 = 0$. That is, one and the same logarithm is obtained for $+1$, -1, and i. *Secondly:* On your assumptions 2^0 has an infinite number of meanings, for 2^0 will equal -1, also $+1$, $\sqrt{-1}$, $\sqrt[4]{-1}$, $\sqrt[8]{-1}$, etc. Unless 2^0 is many-valued, these must all be equal to each other. If $2^0 = +1$, then 2^0 is single-valued and no such difficulty arises. *Thirdly:* If $x^e = -2$, $x^{2e} = +4$, but this transition from $-n$ to $+n$ you yourself reject. *Fourthly:* If $\log(-n)$ is real, then $\log\sqrt{-n}$ is real; hence, impossible numbers would have possible logarithms. The assumption that only $+n$ have logarithms avoids this trouble. *Fifthly:* From the beginning you have admitted that we cannot have $2^e = 1$ and $2^e = -1$ at the same time. But if you put $2^0 = -1$, then $(2^0)^2 = 2^0 = +1$. Hence $2^e = 1$ and $2^e = -1$, for $e = 0$. This is contrary to your admission.

All three things show that your hypothesis concerning $\log(-n)$ is unnatural, useless, and inadmissible. I have shown elsewhere that proportions cannot be formed, involving $-n$. If it be true that the two fractions $+1/-1$ and $-1/+1$ are equal, observe that fractions are not the same as ratios. It is evident from all this that the very foundations of things analytical have been neglected thus far.[1]

June 7, 1713. Bernoulli to Leibniz: What do you understand by *natural?* If that is natural which conforms with usage, then $\log(-n)$ is less natural than $\log(+n)$. The first of your five objections to $\log(-n)$ is that some $+n$, $-n$, in would have the same logarithms. I admit only that $\log(+n) = \log(-n)$. The half of any logarithm is not necessarily the logarithm of the square root; it is rather the logarithm of the mean proportional between $+1$ and $+n$, or -1 and $-n$. The mean proportional between -1 and -1 is $\sqrt{-1 \times -1} = +\sqrt{+1}$ or $-\sqrt{+1}$. There is nothing absurd in this. *Secondly:* I deny that $2^0 = \sqrt{-1} = \sqrt[4]{-1}$, etc. As just explained, $2^0 = \sqrt{-1 \times -1}$ and $= \sqrt{-1 \times -1 \times -1 \times -1}$, etc. All these radicals equal $\sqrt{+1}$ or ± 1. There is no discord in these results. *Thirdly:* You say that if $x^e = -2$, then $x^{2e} = +4$. The logarithmic curve shows this to be untrue. Twice $\log(-n)$ is not $\log n^2$. The third proportional of $-n$ is obtained from $-1 : -n = -n : x$. Hence if $x^e = -2$, then $x^{2e} = -2 \times -2 \div -1 = -4$. Consequently there is no crossing from $-n$ over to $+n$. *Fourthly:* My definition of the mean proportional of $-n$ does not lead to the absurd result that in has a possible logarithm. *Fifthly:* If $2^0 = -1$, then $2^{2 \cdot 0}$ is not $= +1$, but to $-1 \times -1 : -1 = -1$. Hence the absurd result does not follow that 2^0 is at the same time $+1$ and -1.

June 28, 1713. Leipniz to J. Bernoulli: I have no time to disprove your objections to my doctrine which makes $\log i$ impossible, the double of impossibles impossible, $\log n$ the double of $\log\sqrt{n}$. If you assume logarithms in which this is not so, that is nothing to me. I call the more *natural*, not that which is more customary, but that which is nearer to nature and the more simple.

July 29, 1713. J. Bernoulli to Leibniz: You do not deny that the assumption $+1$ is arbitrary, and that -1 is permissible. According to the latter, $\log(-1) = 0$. From this follows all I have previously said about $\log(-n)$.

It is easy to see that Leibniz and J. Bernoulli could not come to an agreement on $\log(-n)$, as long as they did not agree on the definition of "mean proportional" and "third proportional," when applied to negative numbers. There is no need on our part to enter into a minute discussion of the validity of the arguments presented. Bernoulli's argument involving infinite areas between the hyperbola and its asymptotes was repeatedly bombarded during the eighteenth century, but was never hit at its vulnerable point, namely the assumption

[1] *Ex quibus intelligitur, in ipsis rei Analyticæ fundamentis aliqua adhuc neglecta fuisse.*

that $\infty - \infty = 0$.[1] It is interesting to note that logarithms are part of the time connected with the two progressions, as in Napier's definition of a logarithm, but most of the time with the exponential concept as expressed in the exponential notation of the present time. Leibniz makes few appeals to geometry; J. Bernoulli uses curves repeatedly, as if more could be gotten out of a figure than is put into it.

Leibniz insists that log (-1) and log $(\sqrt{-1n})$ do not exist. *Non potest dari Logarithmus* $\sqrt{-2}$. Non-existence is based on inconceivability. We shall see later that Euler puts a different intepretation upon Leibniz; he represents Leibniz as contending that log (-1) is imaginary, not as non-existing. Leibniz died three years after the close of this controversy. This correspondence between him and John Bernoulli during the years 1712 and 1713 was not published until 1745. Not until then did the logarithms of negative numbers engage the attention of mathematicians in general.

Meanwhile there are other researches demanding our attention. In an article in the *Philosophical Transactions* of London, published in 1714, Roger Cotes develops an important formula which in modern notation is

$$i\varphi = \log(\cos\varphi + i\sin\varphi).^2$$

In 1722, after the death of Cotes, this article was republished in his *Harmonia mensurarum*. He introduces as the "measure of a ratio" (mensura rationis) the logarithm of the ratio, multiplied by a constant or modulus. As Braunmühl points out, this "measure of a ratio" was long lost sight of, but was introduced anew in the nineteenth century. We have already called attention to the fact that Cotes himself was anticipated by Edmund Halley in this mode of measuring ratio.

Without stopping to explain how "the measure of the ratio" figures in Cotes's

[1] Proofs involving the comparison with each other of infinite areas in a plane appeal to our intuition with great force. They were accepted as valid by some writers of the nineteenth century as well as of the eighteenth century. Louis Bertrand based upon such comparison his proof of Euclid's parallel-postulate. See L. Bertrand, *Developpement nouveau de la partie elementaire des mathematiques*, T. II, a Geneve, 1778, pp. 19, 20. This proof was accepted by Johann Schultz. See J. Schultz, *Versuch einer genauen Theorie des Unendlichen*, Königsberg und Leipzig, 1788, pp. xi–xv. Similar reasoning was endorsed by Johann Heinrich Lambert. See Lambert, *Deutscher gelehrter Briefwechsel*, Bd. I, Berlin, 1781, p. 118, in a letter dated Feb. 2, 1766. Substantially Louis Bertrand's proof was published anonymously in Crelle's *Journal* in 1834. It was accepted as valid by A. De Morgan. See De Morgan on "Infinity and the Sign of Equality" in *Trans. Cambridge Phil. Society*, Vol. XI, Part I, p. 158. It was translated and published by W. W. Johnson in the *Analyst* (Des Moines, Vol. III, 1876, p. 103). See Cajori, *Teach. u. Hist. of Math. in U. S.*, 1890, p. 379.

[2] That Cotes derived this relation was pointed out by Timtschenko in his History of the Theory of Functions, 1899 [Russian], pp. 519–522. It receives emphasis also in an article by A. v. Braunmühl in *Bibliotheca mathematica*, 3d S., Vol. V, 1904, pp. 355–365. How it happened that so important a theorem should remain in the writings of Cotes for 185 years, without being detected, may perhaps be inferred from Cotes' mode of statement: "Nam sit quadrantis circuli quilibet arcus, radio CE descriptus, sinum habeat CX sinumque complementi ad quadrantem XE; sumendo radium CE pro Modulo, arcus erit rationis inter $EX + XC\sqrt{-1}$ et CE mensura ducta in $\sqrt{-1}$."

derivation of his formula for $i\varphi$, his process may be roughly outlined in modern notation as follows: The area of the surface generated by an arc of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, when revolved around the Y-axis and taken between the limits $y = 0$, $y = y$, can be expressed in two ways, namely,

$$S = a\pi\left\{y\sqrt{1+\frac{a^2-b^2}{b^4}y^2}+\frac{b^2}{\sqrt{a^2-b^2}}\log\left(y\sqrt{\frac{a^2-b^2}{b^4}}+\sqrt{1+\frac{a^2-b^2}{b^4}y^2}\right)\right\},$$

$$S = a\pi\left\{y\sqrt{1-\frac{b^2-a^2}{b^4}y^2}+\frac{b^2}{\sqrt{b^2-a^2}}\varphi\right\},$$

where $\sin\varphi = y\sqrt{\frac{b^2-a^2}{b^4}} = iy\sqrt{\frac{a^2-b^2}{b^4}}$, $\cos\varphi = \sqrt{1+\frac{a^2-b^2}{b^4}y^2}$. Comparing the two expressions for S, we have

$$i\psi = \log\,(i\sin\psi + \cos\psi). \tag{1}$$

While Cotes was anticipated by John Bernoulli I in establishing a relation between the logarithm of an imaginary number and a goniometric function, Cotes, in turn, anticipated the continental mathematicians in the derivation of $i\varphi = \log\,(\cos\varphi + i\sin\varphi)$. It was much later, in a letter of Oct. 18, 1740, that Euler stated to John Bernoulli I that $y = 2\cos x$ and $y = e^{+x\sqrt{-1}} + e^{-x\sqrt{-1}}$ were both integrals of the differential equation $d^2y/dx^2 + y = 0$; that the two integrals were equal to each other, since both could be expanded into the same infinite series. Euler makes remarks from which it follows that he knew at that time the corresponding exponential expression for *sin* x.[1] Both expressions are given by him in *Miscellanea Berolinensia* 1743 and again in his *Introductio in analysin*, Lausannæ, 1748, Vol. I, p. 104, where he gives also the all-important formulæ, $e^{+r\sqrt{-1}} = \cos v + \sqrt{-1}\sin v$, $e^{-r\sqrt{-1}} = \cos v - \sqrt{-1}\sin v$. From Cotes's formula $i\varphi = \log\,(\cos\psi + i\sin\psi)$, given in 1714 and 1722, to Euler's exponential form is an easy step, yet over a third of a century intervened between the first publication of the one and of the other. It is interesting to observe that neither Cotes nor Euler appear to hesitate in, or to recoil from, the use of $\log\,(\cos\psi + i\sin\psi)$, involving the logarithm of complex numbers. Moreover, neither Cotes, nor Euler in his *Introductio,* make any attempt to use this relation in the discussion of the theory of logarithms of complex numbers. Both were aware of the periodicity of the trigonometric functions. Had Cotes applied the idea of periodicity to $i\varphi = \log\,(\cos\psi + i\sin\psi)$ he might have anticipated Euler by many years in showing that the logarithm of a number has an infinite number of different values.

The theory of logarithms of negative numbers was incidentally touched by Euler very early, in his correspondence with John Bernoulli I. The letters which passed between these men in 1727–1731 have been in the possession of the Stockholm academy of sciences and have for the first time been published in full by

[1] *Bibliotheca mathematica,* 2d S., Vol. 11, 1897, pp. 48–49.

G. Eneström in 1902.[1] A translation from the original Latin into German was brought out by E. Lampe.[2] It will be remembered that Euler was a pupil of John Bernoulli I and had followed the two sons (Daniel and Nicolaus) of John Bernoulli to St. Petersburg. Euler was then 20 years old; John Bernoulli was 60. The following is a synopsis of the correspondence, in so far as it bears on logarithms:

Nov. 5, 1727. Euler to J. Bernoulli: The equation $y = (-1)^x$ is difficult to plot, since y is now positive, now negative, now imaginary. It cannot represent a continuous line.[3]

Jan. 9, 1728. J. Bernoulli to Euler: If $y = (-n)^x$, then $ly = xl(-n)$ and $dy/y = dx \cdot l(-n) = dx \cdot l(+n)$; for $dl(-z) = -dz/-z = +dz/+z = dlz$. Integrating, $ly = xln$, and $y = n^x$. Hence $y = (\pm 1)^x$ becomes $1^x = 1$, or $y = 1$.

Dec. 10, 1728. Euler to J. Bernoulli: I have arguments both for and against $lx = l(-x)$. If $lxx = z$, we have $\frac{1}{2}z = l\sqrt{xx}$. But $\sqrt{xx}$ is as much $-x$ as $+x$. Hence $\frac{1}{2}z = lx = l(-x)$. It may be objected that xx has two logarithms, but whoever claims two, ought to claim an infinite number.[4] Argument against: From the equality of the differentials we cannot infer the equality of the integrals. Moreover, $l(-x) = lx + l(-1)$; hence $l(-x) = lx$ only if $l(-1) = 0$. Again, if $lx = l(-x)$, then $x = -x$ and $\sqrt{-1} = 1$, but I rather think the conclusion from the equality of the logarithms to the equality of the numbers cannot be drawn. Your expression for the area of a circular sector[5] of radius a, viz. $aa/(4\sqrt{-1}) \times l(x+y\sqrt{-1})/(x-y\sqrt{-1})$, becomes for a quadrant, x being then 0, $aa/(4\sqrt{-1}) \cdot l(-1)$. Hence, if $l(-1) = 0$, we must have $\sqrt{-1} = 0$ and even $1 = 0$. Most celebrated Sir, what do you think of these contradictions?

April 18, 1729. J. Bernoulli to Euler: When I say that $lx = l - x$, it is to be understood that $l - (x)$ is meant, not $l(-x)$. Thus, $l - (x)^{\frac{1}{2}}$ is real, but $l(-x^{\frac{1}{2}})$ is imaginary. The area of the circular sector is 0, when $x = 0$, however much it ought to be equal to the quadrant. Let the constant Q be a quadrant, then we may write the area of the sector generally $= aa/(4\sqrt{-1})l(x+y\sqrt{-1})/(x-y\sqrt{-1})+nQ$, so that, the first term vanishing when the sector becomes a quadrant, n may be so chosen as to make nQ any multiple or submultiple of the quadrant we need. For a semi-quadrant we have $aa/(4\sqrt{-1})l\sqrt{-1}$, which is 0, since $l\sqrt{-1} = 0$. Here we must take $n = \frac{1}{2}$.

May 16, 1729. Euler to J. Bernoulli: The difference between $l - (x)$ and $l(-x)$ is not clear to me. The expression $4a/(4\sqrt{-1})l(x+y\sqrt{-1})/(x-y\sqrt{-1})$, thought to be constant, appears to me to be increasing, since $x = 0$ exhibits a vanishing sector. That nQ ought to be added, I do not as yet see. If n can be $\frac{1}{2}$, it can be $\frac{1}{4}$ and any number. It would be superfluous to show that $aa/(4\sqrt{-1})l(x+y\sqrt{-1})/(x-y\sqrt{-1})$ must express a sector, if nQ alone were sufficient to represent any sector whatever. Neither of us can afford to run into paralogisms.

In this correspondence between John Bernoulli I and L. Euler, Bernoulli holds the view that $\log n = \log(-n)$. This is the same formula which he defended 16 years earlier, in letters to Leibniz. Now, as then, Bernoulli argues from the equality of two differentials to the equality of the two resulting general

[1] See *Bibliotheca mathematica*, 3d S., Vol. 4, 1902, pp. 344–388. Information on these letters is given also in the same journal, 2d S., Vol. 11, 1897, pp. 51–56; Vol. 13, 1899, p. 46.

[2] *Festschr.* z. Feier d. 200. Geburtstages L. Eulers, Leipzig and Berlin, 1907, pp. 119–137.

[3] In this synopsis we follow the notation used in the letters except in our use of slanting fractional lines and parentheses necessitated by them.

[4] *Posset quidem objici, xx habere duos logarithmos, sed hoc qui asser[ere vult] infinitos adjudicare deberet.*

[5] To facilitate the derivation of this expression, we note that

$$\int (a^2 - x^2)^{-\frac{1}{2}} dx = -i\int (x^2 - a^2)^{-\frac{1}{2}} dx = -i \log(x + \sqrt{x^2 - a^2}) + c$$
$$= -i/2 \log(x+iy)^2/(x^2+y^2) - i/2 \log a^2 + c = -i/2 \log(x+iy)/(x-iy) - i/2 \log a^2 + c.$$

integrals. In his argument on sectorial areas (Apr. 18, 1729) he confuses definite integrals with general integrals. His distinction between $l - (x)^{\frac{1}{2}}$ and $l(-x^{\frac{1}{2}})$ is not made clear. Perhaps he means simply that $l(-x^{\frac{1}{2}})$ shall signify $l(-x)^{\frac{1}{2}}$, when x itself is positive.

Euler's argument, that $e^{z/2} = \pm x$ yields $\log x = \log(-x)$, involves an interesting point. When we write $a^b = c$ and define $b = \log_a c$, a and c are taken to be both single-valued. Euler drops this restriction on c. He takes $e^{z/2} = \pm x$, so that the definition implied in his mode of procedure, viz. $z/2 = \log(\pm x)$, really amounts to two definitions, $z/2 = \log x$ and $z/2 = \log(-x)$. Now there is no objection, *a priori*, to two distinct definitions. In vector analysis we have at least two definitions of multiplication, yielding a vector-product $a \times b$, and a scalar product $a \cdot b$. The question to be considered is, can the two definitions be used side by side? Do the two fit together so as to give a non-contradictory logarithmic theory of complex numbers? The conclusion is drawn from the two definitions and the ordinary rules of operation, that all roots of $+1$ and -1, in other words, all complex numbers of unit modulus, have zero for their logarithm. This is certainly very simple, also quite useless. Previous to the Euler-Bernoulli correspondence no real contradictions had been pointed out in this theory. It was Euler who gave it a death blow by pointing out that $\log \sqrt{-1} = 0$ was in conflict with the formula $\sqrt{-1}\,\pi = 2l\sqrt{-1}$, resulting from J. Bernoulli's accepted expression for the area of a circular sector. The blow dealt by Euler was not considered fatal at the time. Definite integrals were as yet indefinitely comprehended. J. Bernoulli's theory of logarithms continued to find defenders. Euler made another important remark in his letter of December 10, 1728; he touches for the first time the truth that $\log n$ has an infinite number of values. But he does not pursue this matter further at this time.

THE UNION OF THE LOGARITHMIC AND EXPONENTIAL CONCEPTS.

The possibility of defining logarithms as exponents was recognized in the seventeenth century by John Wallis, but not until about 1742 do we find a systematic exposition of logarithms based on this idea. About this time it came to be recognized that involution has two inverses, different in kind, namely, evolution and logarithmation; in the first inverse we assume, in $a^b = c$, b and c as given and find a, in the second inverse we assume a and c as given and find b. Tropfke[1] names William Gardiner as the first to give the new definition of logarithms and to base the theory of logarithms upon it. It is given in the introduction to Gardiner's *Tables of Logarithms*, London, 1742. The definition is as follows: "The common logarithm of a number is the Index of that power of 10 which is equal to the number." It is practically certain that this definition is not due to Gardiner, but to William Jones. "The Explication of the Tables," says Gardiner, ". . . I have collected wholly from the papers of W. Jones, Esq." Maseres,[2] who reprinted this "Explication" in 1791, attributed it entirely to

[1] Tropfke, *op. cit.*, Vol. II, 1903, p. 142, note 576.
[2] Maseres, *Scriptores logarithmici*, Vol. II, London, 1791, p. 1.

Jones. Prof. W. W. Beman informs me that Jones's exposition of logarithms, as given in his *Synopsis palmariorum matheseos*, 1706, was based on Halley's treatment of 1695, but a posthumous paper by Jones, published in the *Philosophical Transactions* for the year 1771, gives the new definition. Whether Jones printed this definition before Gardiner is still undetermined. The one whose influence was greatest in emphasizing the new view was Euler, who, in his *Introductio*, 1748, Chap. VI, §102, gives the definition involving exponents. In this same chapter Euler gives an exposition of negative and fractional exponents and calls attention to the multiple values of a number having a fractional exponent, an explanation seldom found in mathematical treatises of that time. That the new definition of a logarithm was in every way a step in advance has been doubted by some writers. Certain it is that it involves internal difficulties of a serious nature.

[*To be continued.*]

MINIMUM COURSES IN ENGINEERING MATHEMATICS.

By S. EPSTEEN, University of Colorado.

Introduction.—This paper was suggested by a number of inquiries as to the nature and content of the course in engineering mathematics at the University of Colorado. This course is based on three entrance units in Mathematics, and consists of algebra, trigonometry, analytic geometry, the calculus, and least squares as prescribed courses and differential equations, higher calculus, vector analysis, Fourier's series and other advanced courses as electives.

The following outline[1] is not that of a complete course in engineering mathematics, nor even the average course, but, as the title of the paper indicates, the minimum course. The average course is based on this minimum but contains more material, both theoretical and applied. This outline gives an irreducible minimum. A course falling below this standard may be a good trade school course, it may be a most useful and practical course in many respects, but it is not a course in engineering mathematics.

Mathematics and Engineering.—Engineering mathematics is in no sense trade school mathematics or practical arithmetic. A trade school may have little use for mathematics as a science, but the engineering college demands a knowledge of principles as well as facts. This is particularly noticeable in the recent advances in the profession of civil engineering which have been along the lines laid down by Rankine and not by Trautwine. To the engineering student mathematics is as essential as anatomy is to the surgeon, as chemistry is to the apothecary, as drill is to the army officer. The professor of engineering is certainly on firm ground when he takes the stand that the mathematics taught to his students should not be too abstract on the one hand nor too concrete on the other. If the subject matter is too abstract it is unintelligible or uninteresting to the beginner; if it is

[1] The Editors, while in sympathy with the broad purposes of this paper, share no responsibility for the details of the suggested programs.

too concrete the science degenerates to the mere performing of certain mechanical operations—to a common tool instead of a valuable instrument.

Mathematicians and Engineers.—Until very recently there was a complete lack of understanding and sympathy between mathematicians and engineers. In the proceedings of the Society for the Promotion of Engineering Education, volume 17, page 39, the chairman of the joint committee makes the following statement: "The mathematicians expected that the engineers would tell them that what they wanted is practical arithmetic, while the engineers fully expected that the mathematicians would tell them that what they wanted is 'pure mathematics,' the queen of the sciences.' But as the session progressed everyone was astonished to find that no conflict was precipitated." The situation has greatly improved since that time. In volume 19 of the same proceedings we find a complete syllabus of a minimum course in mathematics for engineering students. The report is by an individual, not by the entire committee. While the author's eminence as a mathematician is above all question, his experience with engineering is not as extensive as his experience in pure mathematics. As might be expected under such circumstances, his report is substantially sound, but it needs revision in details. For instance, in the algebra course, no mention is made of graphic algebra. This topic should be included. On the other hand, harmonic progressions are mentioned, with the comment that they are not of great importance. This topic should be excluded.

The Engineering Student and the Arts Student.—The purpose and aim of the prospective engineer are so radically different from that of the general student that the content of the course should be different from that which is generally given to the student who elects mathematics as a part of a general education. Experience proves that the best instruction to engineering students is not given by instructors who have no patience with anything except pure mathematics, but by those who have had some training in engineering or at least considerable training in mathematical physics. That is to say, the most sympathetic instructors are generally those who are near enough to engineers to take some interest in the concrete problems themselves as distinct from their solutions.

The Freshman and Sophomore Years.

Entrance Requirements.—Many high schools are rebelling against college domination. Their stand is well taken. Only a small percentage of their students go to college and it is therefore the obvious duty of the high school to concentrate its chief efforts on the majority. Most engineering colleges demand three units in mathematics for entrance, consisting of one and one-half years of algebra, one year of plane geometry and one-half year of solid geometry. Under the existing circumstances this is all that can be effectively enforced, at least in some parts of the country, even if more is desirable, and even though more is frequently offered.

Engineering Algebra.—While the amount of time devoted to each course differs considerably in the various colleges, the general average for algebra is

three times per week for one semester (or the equivalent); that is, 54 class periods less holidays and quiz days, making a total of 45 class periods. In view of the fact that the course must necessarily begin with a brief review, it is perfectly obvious that the ordinary text book contains far too much material. Many authors argue that this is desirable in that it gives the instructor a wider range of selection and it impresses upon the student the fact that when he has finished the course he has not learned all there is to know about it. Whatever merit or demerit there may be in the second argument, the first is not strictly sound. Even an experienced instructor does not always know what to select from the immense amount of material at his disposal, while the inexperienced instructor is quite at sea. By inexperienced is meant not merely those who have done little or no teaching, but also those whose interests are exclusively in pure mathematics and who have had no practical experience with engineers, their points of view and their standards of value. For example, even experienced instructors frequently select theory of equations as a suitable topic, whereas, in reality, this chapter is relatively unimportant to an engineer as compared with topics that are omitted when this one is included. An actual inquiry shows that the professors of electrical engineering and mechanical engineering place almost no value on this topic; and the professors of civil engineering state that while they occasionally want the students to know how to solve a cubic or a quartic equation, the need is so rare that they would willingly forego this topic for others.

The minimum course in algebra should consist of the following chapters: (1) Review of fractional and negative exponents, reduction of surds, imaginaries, zero, infinity; (2) Logarithms; (3) Variables, functions, graphs; (4) Simultaneous linear equations, determinants of the second and third orders, graphical solution; (5) Quadratic equations; (6) Simultaneous quadratics; (7) Variation; (8) Arithmetic and geometric progressions (number of terms finite, infinite); (9) Binomial theorem, the series for e and e^x; (10) Complex numbers, vectors, addition, subtraction, multiplication and division; (11) Partial fractions; (12) Permutations, combinations, probability. There are about forty formulas in these chapters which should be understood and memorized so that they may become the student's everyday working tools—his mental machinery. There should be numerous applications to simple problems taken from actual engineering practice.

Engineering Trigonometry.—In trigonometry the problem of a suitable course is much simpler than in algebra. The general average of time allowed to this course is two hours per week for one semester or the equivalent; that is, 36 days less holidays and quiz days, making about 30 class periods. Trigonometry is needed primarily for three specific purposes—surveying, physics, calculus. The minimum course can therefore be summarized in about forty formulas under the following headings: (1) Trigonometric functions; (2) Right triangles; (3) The addition theorems; (4) Inverse trigonometric functions, trigonometric equations; (5) The oblique triangle; (6) Circular measure, graphical representation. If the students have had logarithms early in the algebra course, so that this topic may be omitted in the trigonometry, it is possible to

get in a short chapter on the right spherical triangle. The students of civil engineering need spherical trigonometry in some of their other courses. A brief study of the right spherical triangle is not enough, but it constitutes an introduction, and this is all that can be reasonably expected under the time limitations.

Engineering Analytics.—Most colleges devote five hours per week for one semester to analytic geometry; that is, about 80 actual class periods. While one cannot cover this subject fully in that time, it is sufficient for a fair introduction. Indeed, if eighty periods could be given to algebra also, instead of forty-five, the results would be large in proportion to the extra time. Analytic geometry is an indispensable part of any good mathematical curriculum. Although highly interesting in itself, it would be a still better course if frequent applications were introduced. The newer books on algebra, trigonometry and calculus abound in interesting and important applications, but not so in analytic geometry. Authors of the text books on this subject still follow slavishly in the footsteps of their predecessors and concrete applications are exceedingly rare in a book on analytics.

The minimum course should consist of about seventy-five formulas under the following headings: (1) The straight line; (2) The circle; (3) The parabola; (4) The ellipse; (5) The hyperbola; (6) Polar coordinates; (7) Transformation of coordinates; (8) Higher plane curves; (9) Coordinates in space. It is well to take up a chapter on the general equation of the second degree if time permits, but this is by no means an essential part of the minimum course, as the author of the syllabus (Art. 5) thinks. Indeed, a chapter on higher plane curves, including the cycloid, hypocycloid, epicycloid, involute and others, is far more important.

Engineering Calculus.—Calculus is doubtless the most valuable and effective instrument the engineer can possess. Of all the subjects in the curriculum no other is so far reaching and powerful in its applications. Still, one sometimes hears a successful engineer say "I use algebra and trigonometry a great deal but never the calculus." Whenever a man makes this assertion, it usually indicates that he had defective instruction in the calculus when he first took the course. Naturally, he could not use an instrument which he did not understand. He may have been a good engineer without the calculus, but he would have been a better engineer with it.

A minimum course in the calculus should consist of a thorough understanding of the proofs and applications of about one hundred formulas and theorems included in the following chapters: (1) Functions, continuity, limits (a very brief chapter); (2) Differentiation; (3) Simple applications to geometry, physics, mechanics, maxima and minima; (4) Integration as anti-differentiation; (5) Integration as the limit of a sum, definite integrals; (6) Applications of integration; (7) Centers of mass, moments of inertia, radii of gyration (special emphasis on plane sections); (8) Series, indeterminate forms; (9) Approximate methods of integration; (10) Differential equations with applications. All the important formulas and theorems should be memorized.

The Junior Year.

Least Squares.—The subject of least squares should be required of all students in civil engineering, and probably of those in mining engineering, in the first half of the junior year. Differential equations should be offered as an elective in the second half year.

The minimum course in least squares should consist of about forty formulas under the following headings: (1) Review of theory of probability; (2) Errors and their probability, probability curve, probability integral; (3) Important processes such as solution of observational equations (equal and unequal weights), average error, percentage error, mean error, approximation formulas, propagation of errors; (4) Applications to civil engineering and physics; (5) Normal equations; (6) Fitting formulas to observations.

Many mathematicians intensely dislike least squares and do not want to teach the course. The sentiment seems to be founded on the fact that the ideas and methods of least squares are foreign to a mathematician's usual mode of thought and standards of judgment. The subject does not belong to the domain of logical mathematics, and it is thus different from the usual course in pure mathematics. It belongs to the domain of experimental mathematics, and when viewed in the right light, is really fascinating.

Differential Equations.—The subject of differential equations should be required of all students in electrical, chemical and mechanical engineering in the second half of the junior year. Least squares should be offered as an elective in the first half of the year. In a three hour course the following ground can be covered: (1) Formation of differential equations; (2) Equations of first order and first degree; (3) Equations of first order but not of first degree; (4) Singular solutions; (5) Applications to physics, mechanics, geometry; (6) Linear equations of second order with constant coefficients, with applications; (7) Linear equations with variable coefficients, with applications; (8) Equations of particular forms; (9) Miscellaneous applications.

The Senior Year.

Advanced Courses.—The courses for seniors and graduates should be elective. Almost any advanced course may be offered, although it is well to emphasize those which have some connection with applications. Some of the most important courses are: (1) Determinants and theory of equations; (2) Advanced calculus; (3) Fourier's series; (4) Bessel's functions; (5) Spherical and cylindrical harmonics; (6) Vector analysis; (7) Elliptic functions; (8) Theory of functions; (9) Partial differential equations; (10) Linear differential equations.

Advanced Calculus.—Very few departments of mathematics are large enough to offer all the foregoing courses each year. The best that can be expected is to have one or two each semester. Advanced calculus is, in a sense, one of the most important in the list and is worthy of special mention. The minimum course for engineers should include short chapters of an introductory nature on: (1) Infinite series; (2) Definite integrals; (3) Hyperbolic functions: (4) Elliptic functions, Bessel's functions; (5) Beta and gamma functions; (6) Fourier's series.

Conclusion.—Each branch of engineering demands a somewhat different mathematical training from every other branch. For this reason every text book and every course must necessarily be a compromise. There are on the market some very acceptable books on algebra, trigonometry and the calculus. The pressing need is for equally acceptable books on analytic geometry, least squares and differential equations. There are, indeed, some very good American texts on differential equations, but, with possibly one exception, it does not appear that their authors have made any particular effort to find out which chapters are most useful to engineers, or to provide applications adapted to the needs of engineers, beyond certain standard types well known to all mathematicians.

A GEOMETRIC INTERPRETATION OF THE FUNCTION F IN HYPERBOLIC ORBITS, CORRESPONDING TO THAT OF THE ECCENTRIC ANOMALY E IN ELLIPTIC ORBITS.[1]

By W. O. BEAL, Illinois College.

The Geometric Interpretations of E and M in Elliptic Orbits.

If the elements of an elliptic orbit of a planet are given in the problem of the relative motion of two bodies, subject only to their mutual gravitation, the polar coördinates $r = SP$ (Fig. 1) and $v = \angle ASP$, the radius vector and true anomaly of the planet, can be computed for any time t in the following manner:

Compute the mean daily motion

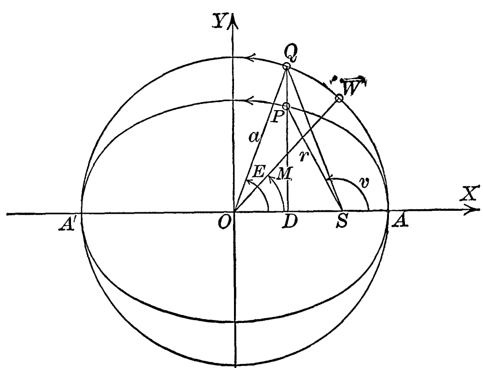


Fig. 1.

$$n = \frac{k\sqrt{m_1 + m_2}}{a^{\frac{3}{2}}},$$

[1] This problem is given by F. R. Moulton in his *Introduction to Celestial Mechanics*, p. 159.

the mean anomaly $M = n(t - T)$, the eccentric anomaly E, by solving Kepler's equation

$$E - e \sin E = M,$$

and then $r = a(1 - e \cos E)$, and

$$\tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}.$$

In these, a, T and e stand for the semi-major axis, time of perihelion passage and eccentricity of the relative orbit. The function E is first introduced into the problem analytically to aid in the integration of the equation of areas. It has the well known geometric meaning that it is always equal to the central angle AOQ, in the auxiliary circle, which corresponds to the true anomaly v.

We may give a geometric meaning to M by imagining a point W moving with uniform speed along the auxiliary circle in such a way that it would be at the perihelion point A of the orbit every time the planet was there. Thus M would be the central angle AOW described by the radius vector of this point.

It is further true that the area of the

$$\text{sector } AOQ = \tfrac{1}{2}a^2E,$$

and

$$\text{triangle } OSQ = \tfrac{1}{2}a \cdot ae \sin E.$$

Hence

$$\text{Area } ASQ = \tfrac{1}{2}a^2(E - e \sin E) = \tfrac{1}{2}a^2 M$$

$$= \text{Area sector } AOW.$$

Thus M and E are proportional to the areas generated by SQ and OQ respectively, and would be numerically equal to twice those areas if $a = 1$.

The Geometric Interpretations of F and M in Hyperbolic Orbits.

The radius vector and true anomaly, r and v, in a hyperbolic orbit can be found in like manner for any instant t by the formulas,

$$(1) \quad \begin{cases} n = \dfrac{k\sqrt{m_1 + m_2}}{a^{\frac{3}{2}}}, \\ M = n(t - T), \\ -F + e \sinh F = M, \\ r = a(-1 + e \cosh F), \\ \tan \dfrac{v}{2} = \sqrt{\dfrac{e+1}{e-1}} \tanh \dfrac{F}{2}. \end{cases}$$

Like E, the function F is first introduced analytically as an auxiliary quantity.

We are to find geometric meanings for F and M analogous to those given to E and M in an elliptic orbit. For this purpose use the rectangular hyperbola having the same center and semi-major axis a as the orbit, and call it the auxiliary hyperbola. Also use the auxiliary circle with radius a and center at the center of the orbit. Thus in Fig. 2, $P'AP$ represents a hyperbolic orbit, with focus at S, perihelion at A, center at O, and P any position on the orbit; also $Q'AQ$ is the auxiliary hyperbola, with Q corresponding to P.

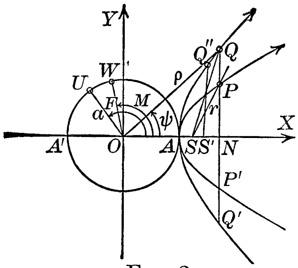


FIG. 2.

Join Q to O, and call $\angle\, NOQ = \psi$, $OQ = \rho$. Then the equation of the auxiliary hyperbola in rectangular coördinates is

$$x^2 - y^2 = a^2, \tag{2}$$

which expressed in polar coördinates gives $\rho^2(\cos^2\psi - \sin^2\psi) = a^2$, or

$$\rho^2 = \frac{a^2}{\cos 2\psi}. \tag{3}$$

Next compute the area bounded by OA, OQ and arc AQ.

$$OAQ = \tfrac{1}{2}\int_0^\psi \rho^2 d\psi = \frac{a^2}{2}\int_0^\psi \sec 2\psi d\psi = \frac{a^2}{4}\log\tan\left(\frac{\pi}{4}+\psi\right) = \frac{a^2}{2}K, \tag{4}$$

where we place

$$K = \tfrac{1}{2}\log\tan\left(\frac{\pi}{4}+\psi\right). \tag{5}$$

From this we readily deduce

$$\tan\psi = \tanh K. \tag{6}$$

Now the rectangular coördinates of Q may be expressed in terms of a parameter G by the relations

$$\begin{cases} x = ON = a\cosh G, \\ y = NQ = a\sinh G, \end{cases} \tag{7}$$

for these expressions satisfy (2). But

$$\tan\psi = \frac{NQ}{ON} = \frac{a\sinh G}{a\cosh G} = \tanh G. \tag{8}$$

Hence by (6)

$$K = G. \tag{9}$$

Again the area of the triangle

$$OSQ = \tfrac{1}{2}OS \cdot NQ = \tfrac{1}{2}ae \cdot a\sinh G. \tag{10}$$

Hence the area

$$ASQ = \tfrac{1}{2}a^2(e\sinh G - G). \tag{11}$$

Remembering that the changes in G are due to the motion of P, we have by comparison of (11) with (1)

$$G = F, \tag{12}$$

and

$$ASQ = \frac{a^2}{2} M; \tag{13}$$

also (4) becomes

$$OAQ = \frac{a^2}{2} F. \tag{14}$$

Thus M and F are proportional to the areas generated by SQ and OQ respectively, and would be numerically equal to twice those areas if $a = 1$.

But $M = n(t - T)$, that is, M is proportional to the time, from which it follows that as P describes its hyperbolic orbit, the corresponding point Q moves on the auxiliary hyperbola so that the law of areas holds with respect to the focus of the orbit.

From the form of the expressions (13) and (14) which represent the areas ASQ and OAQ, it is apparent that those areas are equivalent to circular sectors in a circle with radius a and central angles M and F respectively. We can thus imagine two points W and U traveling on the circle ARA', so related to the planet P that when P passes the perihelion point A, they will also pass through A. One of them, W, will go around the circle at a uniform speed, with its radius vector describing the area $\frac{a^2}{2} M =$ area ASQ. The other, U, will move around the circle at a non-uniform speed, with its radius vector OU describing the area $\frac{a^2}{2} F =$ area OAQ.

Now from (1)

$$\begin{cases} \dfrac{dM}{dt} = \dfrac{k\sqrt{m_1 + m_2}}{a^{\frac{3}{2}}}, \\[2ex] \dfrac{dF}{dt} = \dfrac{k\sqrt{m_1 + m_2}}{a^{\frac{3}{2}}(e \cdot \cosh F - 1)}, \end{cases} \tag{15}$$

which show that for $e < 2$ the non-uniformly moving point U would be moving faster than W at A; that they would be moving with equal speeds when

$$\cosh F = \frac{2}{e},$$

and thereafter the uniformly moving point W would move the faster.

By placing $F = M$ in (1) we find that W would overtake U at the time determined by

$$\sinh M = \frac{2M}{e}.$$

(15) also shows that the speed of the non-uniformly moving point would always decrease but not be zero. From (6) it follows, since $K = F$, that in an indefinitely great period of time this point U, as well as W, would make an indefinitely great number of revolutions about the circle.

The law of areas for hyperbolic motion gives the sector $ASP = \frac{a^2}{2}\sqrt{e^2 - 1}\,M$.

For a rectangular hyperbola $e = \sqrt{2}$ which substituted in this gives $\frac{a^2}{2}M$, or (13). Hence it follows that the area described by SQ is equal to the area described in an equal time by the radius vector $S'Q''$ of a planet Q'' having the same mass and major axis as the planet P, but moving in an orbit which is a rectangular hyperbola.

THIRD CLEVELAND MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

As many readers of the MONTHLY may be unfamiliar with the history and the aims of the American Association for the Advancement of Science, the following data relating to these subjects may be of interest. In 1847 the Association of American Geologists and Naturalists agreed to resolve itself into the American Association for the Advancement of Science, and decided to hold the first meeting under the new management at Philadelphia in 1848.

At this first meeting of the Association its objects were stated to be as follows: "The objects of the Association are, by periodic and migratory meetings, to promote intercourse between those who are cultivating science in different parts of the United States, to give a stronger and more general impulse, and a more systematic direction to scientific research in our country, and to procure for the labors of scientific men increased facilities and a wider usefulness."

These objects have been modified only slightly during the sixty-four years since the Association was formed. For instance, the term United States has been replaced by the more general term America, and several of the meetings of the Association have been held in Canada. While most of the meetings have been in the east, the migrations have extended as far west as Denver and as far south as New Orleans. Four meetings have been held in each of the two cities Buffalo and Washington; and three in each of the four cities, Boston, New York, Philadelphia, and Cleveland. A few other cities have entertained the Association twice, but in most cases only a single visit to a given city has been made.

Similar organizations exist in many other countries. The British Association, organized in 1831, seems to occupy an especially prominent place in the scientific activity of that country and in the extent of its migrations. Its recent visits to South Africa and to Winnipeg, Canada, were marked with great liberality on

the part of the countries visited; and, according to *Nature*, the Commonwealth of Australia has already granted £15,000 to be handed to the British Association to cover the passages of not fewer than 150 British and foreign men of science, who will attend the meeting of the British Association in Australia in 1914.

While the American Association has thus far confined its migrations to much narrower limits than its prototype, it has already had under consideration meetings on the Pacific Coast as well as in the Hawaiian Islands, and with the growth of scientific activity, it also will doubtless further extend its sphere of influence, by direct contact, to other parts of the country. It has now a membership of more than 8,000 and most of the leaders of American science have been found on its register since its organization.

The scientific work of the Association is conducted by its eleven sections and by the affiliated national societies. About twenty-five of the latter met at Cleveland and their programs formed the most important feature of this meeting. Those which are of especial interest to mathematicians are as follows: The American Mathematical Society, including the Chicago Section; The Astronomical and Astrophysical Society of America; The American Physical Society; The American Federation of Teachers of the Mathematical and the Natural Sciences.

Section A of the Association is devoted to Mathematics and Astronomy. "As it is the policy of the Association to avoid competition with programs presented before special national societies," and as both of the subjects included in the scope of Section A were represented by national societies, this Section arranged for only one session, which was held on Tuesday afternoon, December 31, 1912. During this session the addresses of the retiring vice-presidents of Sections A and B were presented.

In the absence of the retiring vice-president and chairman of Section A, Professor E. B. FROST, Director of Yerkes Observatory, his address, entitled "The spectroscopic determination of stellar velocities, considered practically," was read by Professor J. A. PARKHURST. This address will be published in *Popular Astronomy*. The retiring vice-president and chairman of Section B Professor R. A. MILLIKAN, University of Chicago, presented an address entitled "On unitary theories in physics," which has appeared in *Science*.

In addition to these two addresses the following four papers were presented during the given session of Section A:

(1) Henri Poincaré as a mathematical physicist. By Professor A. G. WEBSTER, Clark University.

(2) Some general aspects of modern geometry. By Professor E. J. WILCZYNSKI, University of Chicago.

(3) Cosmical magnetic fields. By Dr. L. A. BAUER, Director of the Department of Terrestrial Magnetism, Carnegie Institution of Washington.

(4) Preliminary note on an attempt to detect the general magnetic field of the sun. By G. E. HALE, Director of Mt. Wilson Observatory.

In the absence of Professor Hale his paper was presented by Dr. BAUER.

The others were presented by their respective authors. The programs of the affiliated societies were, in general, very much more extensive than those of the Sections. For instance, the program of the American Mathematical Society contained fifty-four titles and the American Physical Society had an almost equally extensive list.

While such meetings afford excellent apportunities to learn important facts which have just been discovered, they serve also to inspire one to renewed efforts, and to extend and deepen friendships between those who are working along various scientific lines. The latter desirable ends are greatly promoted by numerous dinners and smokers as well as by the fact that a large number of those who attend select the same hotel.

The weekly journal called *Science* is the official organ of the Association and the reader is referred to its columns for reports of the meetings of all sections. In some cases, more technical details, as regards the scientific results, may be found in the journals published by the various scientific societies which meet in affiliation with the Association. The next meeting of the Association is to be held at Atlanta, Georgia, under the presidency of Professor E. B. WILSON, Columbia University. Director Frank Schlesinger, Allegheny Observatory, and Professor F. R. Moulton, University of Chicago, were elected chairman and secretary respectively, of Section A. The term of the former office is only one year while that of the latter is five years.

G. A. MILLER,
Retiring Secretary of Section A.

MAXIMUM PARCELS UNDER THE NEW PARCEL POST LAW.

The size of a parcel that can be mailed under the rules of the Parcel Post Law is limited in three ways. It must be not more than three feet and six inches in length. It must not weigh more than eleven pounds. The sum of its length and girth must not exceed six feet. The girth is measured around a cross section perpendicular to the length; and since the greatest girth is the one measured, all such cross sections should be equal to the largest of them if it is desired that the volume be a maximum. Furthermore, since the circle has a larger area for a given perimeter than any other closed curve, the parcel should be in the form of a right circular cylinder if the volume is to be a maximum. Let R denote the radius of the base of the cylinder, V its volume and x its altitude or length. Then $V = \pi R^2 x$, and $2\pi R + x = 6$. From these two equations it is easily found by differential calculus that for a maximum value of V the length must be two feet and the girth four feet. The maximum value of V is 2.546 cubic feet. This is the largest mailable volume.

If the parcel to be mailed has its cross sections all equal and of a specified shape which is not circular, it is still true that the length should be two feet and

the girth four feet for maximum volume. To prove this, let P denote the perimeter of a cross section and A its area. As P changes in value, the shape of the cross section remaining constant, A varies as P^2. That is, $A = kP^2$. Then if L denote the length of the parcel, and V its volume, $V = kP^2L$ and $L + P = 6$. From these two equations the value of P that makes V a maximum is found to be 4. The corresponding value of L is 2.

Since a square has a larger area than any other rectangle having the same perimeter, the largest mailable box in the form of a rectangular parallelopiped is therefore one whose dimensions are $1 \times 1 \times 2$ feet. Its volume is 2 cubic feet. The largest mailable cube is one whose edge is 14.4 inches long. Its volume is 1.728 cubic feet. The largest mailable sphere has a diameter of 17.38 inches. Its volume is 1.592 cubic feet.

A cube whose edge is 15 inches is not mailable because the sum of its length and girth is 75 inches. But a right circular cylinder, whose length is 24 inches and whose diameter is 15.27 inches, is mailable, because the sum of its length and girth is only 72 inches, in spite of the fact that its volume exceeds that of the cube by more than 1,000 cubic inches, and the fact that it would take up more room in a mail car. Thus the use of the sum of length and girth as a measure of size leads to the absurd result that sometimes the smaller of two parcels will be rejected by the postal authorities because it is too large, while the larger of the two will be accepted as being small enough. In spite of this absurd result, the rule can be amply justified on the ground of the ease with which it can be applied. With a six-foot tape, the postal clerk can tell in a moment whether or not a parcel is too large to be mailed. The man who is waiting his turn at the post office window will be glad that the clerk does not have to compute the cubical content of every parcel presented.

W. H. Bussey.

NEW BOOKS.

W. H. Bussey, Chairman of the Committee.

A short account of the history of mathematics, 5th edition, by W. W. R. Ball. London and New York, Macmillan, 1912. 24 + 536 pp. $3.25.

Practical geometry and graphics; a text book for students in technical and trade schools, evening classes, and for engineers, artisans, draughtsmen, architects, etc., by E. L. Bates and F. Charlesworth. New York, Van Nostrand, 1912. 8 + 621 pp. $2.00.

Practical mathematics, by E. L. Boter and F. Chorlesworth. New York, Van Nostrand, 1912. 8 + 513 pp. $1.50.

Elements of descriptive geometry, by G. F. Blessing and L. A. Darling. N ew York, Wiley, 1912. 14 + 219 pp. $1.50.

The method of Archimedes recently discovered by Heiberg; a supplement to "The works of Archimedes," 1897, edited by T. L. Heath. New York, Putnam, 1913. 51 pp. $0.75.

AN INTRODUCTION TO MATHEMATICAL PHYSICS, by R. A. Houston. New York, Longmans, 1913. 9 + 199 pp. $2.00.

MEMORANDA MATHEMATICA; a synopsis of facts, formulæ, and methods in elementary mathematics, by the head master of the Kingswood School, Bath, England. Oxford, University Press, 1912. 4 + 272 + 28 pp. $1.75.

LIFE ASSURANCE PRIMER; a textbook dealing with the practice and mathematics of life insurance, for advanced schools, colleges, and universities. 3d edition, revised and enlarged, by H. Moir. New York, Spectator Co., 1912. 7 + 230 pp. $2.00.

PRACTICAL MATHEMATICS; being the essentials of arithmetic, geometry, algebra, and trigonometry, in 4 parts. Parts 1, 2, by C. I. Palmer. New York, McGraw-Hill Co., 1912. $0.75 each.

GEOMETRICAL OPTICS, by A. S. Percival. New York, Longmans, 1913. 6 + 132 pp. $1.50.

A COURSE IN THE PRINCIPLES OF MECHANICAL DRAWING, by J. B. Whitmore. Columbus, O., Champlin Press, 1913. 2 + 2 + 72 pp. $1.85.

RELIABLE INTEREST TABLES, by E. V. Williams and G. G. Bodeen. Fresno, Cal., Bankers Novelty Co., 1912. $3.50.

BOOK REVIEWS.

New Analytic Geometry. By PERCEY F. SMITH, and ARTHUR SULLIVAN GALE. Ginn and Company, Boston, 1912. x+342 pages.

This new book on analytic geometry is considerably smaller than the *Elements of Analytic Geometry* by the same authors. Much of it follows, word for word, the old book, but many parts have been rewritten and rearranged. The book has been made smaller by shortening the work on the conic sections and by omitting the chapters on euclidean transformations, inversion, and poles and polars. The conics are defined separately, the parabola in terms of focus and directrix, and the ellipse and hyperbola in terms of their foci alone. They are studied by means of their equations in rectangular coordinates rather than by means of polar coordinates as was done in the beginning of the chapter on conics in the old book.

A special feature of the book is the new chapter on "Empirical Equations." The finding of equations to represent, at least approximately, curves given by experiment is not discussed in the current books on analytic geometry, although it is one of the most important parts of the subject for those who use analytic geometry in physics, experimental psychology, and statistics. Eleven pages are given to it in this book.

Another new chapter is on "Functions and Graphs." It is mostly a study of maxima and minima of functions without the use of the calculus. Practical problems in maxima and minima are given to be solved experimentally by means of carefully drawn graphs.

The work on transcendental curves and equations in rectangular coordinates is more extensive than in the old book and has been made into a separate chapter. It includes the graphical solution of transcendental equations, and some tables to lighten the work of computation. The authors say truly in the preface that a student loses interest in a function if he cannot calculate rapidly its numerical values. It is for this reason that they have put in the first chapter tables of squares and cubes, square and cube roots, and three-place tables of logarithms and trigonometric functions.

The book is considerably larger than the *Introduction to Analytic Geometry* by the same authors.

W. H. Bussey.

Mathematical Recreations and Essays. Fifth Edition. By W. W. Rouse Ball. Macmillan and Co., London, 1911. xvi+492 pages.

A great deal of new matter has been added to this interesting book since it was first published in 1892. The fifth edition contains almost 250 pages more than the first and about 100 pages more than the fourth. The work on "Kirkman's School-Girls Problem" has been enlarged and made into a separate chapter. There is a paragraph on the same problem as proposed independently by J. Steiner in a somewhat more general form. There is a new chapter of 20 pages on "The Parallel Postulate," and one of 6 pages on the "Insolubility of the Algebraic Quintic." Those who amused themselves in their youth by making figures known as *Cat's Cradles* by twisting on the hands a loop of string will be interested in the new chapter on "String Figures." The subject is more extensive than most people think. The chapter is 32 pages long and is not supposed to be a complete discussion. It is only indirectly connected with mathematics. The author explains the presence of it and the older chapters on "Astrology" and "Ciphers" by saying that he deliberately gave the book a title which would allow him a free hand to write what he liked.

The parts of the book which are not new have been revised. In the chapter on mechanical recreations, after a discussion of the cut on a tennis ball and the spin on a cricket ball, the author has put in a paragraph on the flight of golf balls. In the chapter on matter and ether theories, he has added a page on the principle of relativity. These are typical instances of the way in which the book has been brought up to date.

W. H. Bussey.

An Elementary Treatise on Cross-Ratio Geometry, with Historical Notes. By Rev. John J. Milne. Cambridge University Press, 1911. xxiii+288 pp.

It is well known that our literature on secondary mathematics is too limited. The ambitious teacher of secondary mathematics, who reads English only, does not possess as good facilities for broadening his knowledge as do his German and French colleagues. Recently there has been considerable improvement along this line, and the volume before us is another step in the right direction.

The author is a classical scholar as well as a mathematician, and he has enriched his work by many historical notes which tend to make it much more attractive and helpful to teachers of mathematics. For the most part these notes are not reproduced from well-known histories, as is too often the case, but they generally bear the impress of originality and of independent study. This feature enhances greatly their value and their attractiveness. The same impression is gained in regard to other parts of the work, making the book valuable to the scholar as well as to the student who may approach it without a knowledge even of the meaning of cross-ratio.

Many readers might have preferred the term *Anharmonic Ratio* instead of *Cross-Ratio* in the title, as the former is more widely used. For instance, it is used almost exclusively in France, and it is in common use in Germany, Italy, and in the English-speaking countries. In view of these facts, we may perhaps not agree with the footnote on page 2, which states that the term cross-ratio "is now generally adopted," unless this reference is supposed to apply to Great Britain only. The term anharmonic ratio is not free from objection, since it is sometimes used in two senses—as including or as excluding harmonic ratio.

In these days of emphasis on things which are closely related to the larger modern mathematical theories, it becomes especially interesting to meet a work in which the emphasis is placed on closer contact with the old Greek mathematicians, and with their methods. It is true that even in the present work attention is frequently directed to the great improvements which have been made regarding methods employed by the Greeks in the use of anharmonic ratios, but these improvements are generally in line with Greek methods.

This close contact with the spirit of the Greeks has its disadvantages. For instance, the treatment of the matter on pages 5 and 9 would have gained considerably, both in clearness and in generality, if elementary notions of groups had been introduced; showing that we are dealing here with the operations of subtracting from unity and of dividing unity, and that the successive steps of this kind give rise to six distinct operations, which constitute a very common group. It might also have been observed that these operations are special cases of more general operations which give rise to the same group.[1]

By exhibiting such contact and by putting more emphasis on the underlying abstract notions it is not only possible to make the presentation more interesting for the mature student but it is also possible to prepare the way for further advances. The days of the special geometric methods of the Greeks are passing, and the more powerful and more comprehensive methods of modern mathematics are rightly coming more and more to the front. It is, however, true that the consistent use of such a fundamental concept as that of anharmonic ratio is a great step in advance of the older and more special geometric methods of the Greeks.

Almost half of the book, Chapters I–X, is devoted exclusively to a treatment

[1] Cf. "Groups of Subtraction and Division," *Quarterly Journal of Mathematics*, vol. 37 (1906), p. 80.

of the point and a straight line, and in this part the reader is not assumed to have any knowledge of geometry beyond the fundamental properties of similar triangles and ratio. The teacher of elementary geometry will therefore find no trouble in reading this part, and he cannot fail to derive from such a reading many new points of view and a deeper sense of the richness of geometry and the beauty of geometric results.

The second part of the book, beginning with Chapter XI, page 130, is devoted to a study of conic sections. A few elementary theorems in geometrical conics are here assumed, as the author believes that time can be saved by getting a working knowledge of the elements of conic sections from the ordinary textbooks before the student is introduced to the theorems of Pascal, Brianchon, Desargues, and others. Figures are given with almost all the theorems and many full solutions are worked out in the text.

Very little use is made of projection and the principle of duality is avoided, as the author desired to exhibit the use of the theory of anharmonic ratio by the direct demonstration of correlative theorems. The book is provided with a somewhat full table of contents, and also with a good index. A number of blank pages for notes are found at the end of the volume. As a whole, the book can be regarded as a thoroughly good piece of work and one which should be in the hands of all teachers of elementary geometry.

G. A. Miller.

PROBLEMS AND QUESTIONS.

B. F. Finkel, Chairman of the Committee.

PROBLEMS FOR SOLUTION.

ALGEBRA.

383. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

Find accurately to 6 decimals, $1.000000854^{1283451}$.

384. Proposed by H. C. FEEMSTER, York, Neb.

A man addressed n envelopes and wrote n checks in payment of n bills. Show that the number of ways of enclosing within each envelope one bill and one check in such a manner that in no instance all the enclosures shall be correct is

$$n!\left\{n! - \frac{(n-1)!}{1!} + \frac{(n-2)!}{2!} - \cdots + (-1)^n \frac{0!}{n!}\right\},$$

taking $0! = 0$.

385. Proposed by J. F. LAWRENCE, Stillwater, Okla.

Show that, if p is prime and > 3, $(2p)! - 2(p)!(p)!$ is divisible by p^5.

386. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given the sequence, $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \cdots, u_1, u_2, u_3, \cdots$. Show that

$$\left.\frac{\sum_1^n u_i}{n}\right]_{n \doteq \infty} = \frac{1}{2}.$$

GEOMETRY.

412. Proposed by S. LEFSCHETZ, The University of Nebraska.

To inscribe in a given circle an isosceles triangle in which the base plus the altitude is equal to a given length.

413. Proposed by D. F. KELLY, New York City.

To construct a triangle, having given the base, vertical angle and the ratio of the altitude to the difference of the other two sides.

414. Proposed by H. C. FEEMSTER, York, Neb.

To construct the cyclic quadrilateral, having given its four sides.

CALCULUS.

334. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Solve the differential equation:

$$\frac{\partial^2 T}{\partial u \partial v} + \frac{2v}{u^2 + v^2 + 1} \frac{\partial T}{\partial u} + \frac{2u}{u^2 + v^2 + 1} \frac{\partial T}{\partial v} = 0.$$

335. Proposed by W. R. LEBOLD, Cambridge, Ohio.

Let $\rho = F(\theta, \phi)$ be the equation in polar coordinates of a closed surface. Show that the volume of the solid bounded by the surface is equal to the double integral.

$$\tfrac{1}{3} \int\int \rho \cos \gamma \, d\sigma$$

extended over the whole surface, where $d\sigma$ represents the element of area, and γ the angle which the radius vector makes with the exterior normal.

[Goursat-Hedrick, *Analysis*, p. 325, ex. 9.]

336. Proposed by EVA S. MAGLOTT, Ada, Ohio.

If a right cone stands on an ellipse, prove that its superficial area is

$$\frac{\pi}{2}(OA + OA')(OA \cdot OA')^{\frac{1}{2}} \sin \alpha,$$

where O is the vertex of the cone, A and A' the extremities of the major axis of the ellipse, and α is the semi-angle of the cone.

MECHANICS.

271. Proposed by B. F. FINKEL, Springfield, Mo.

A hollow spherical shell is filled with a frictionless fluid and rolls down a rough inclined plane. After rolling t seconds, the fluid suddenly solidifies. Determine the subsequent motion of the spherical shell.

272. Proposed by J. F. LAWRENCE, Stillwater, Okla.

A perfectly rough circular cylinder is fixed with its axis horizontal. A sphere is placed on it in a position of unstable equilibrium, and projected with a given velocity parallel to the axis of the cylinder. If the sphere be slightly disturbed in a horizontal direction perpendicular to the direction of the axis of the cylinder, determine at what point the sphere will leave the cylinder.

273. Proposed by F. P. MATZ, Reading, Pa.

A person is placed on a perfectly smooth surface. How may he get off?

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

187. Proposed by E. T. BELL ,New York, N. Y.

If m is any integer, P the product of all the distinct prime factors of m and λ their number, and if $N(x)$ denote the number of divisors of x, then

$$6^{\lambda}\Sigma N(d) \cdot N\left(\frac{m}{d}\right) = N(m) \cdot N(Pm) \cdot N(P^2m),$$

where the sign of summation extends to all divisors of m.

188. Proposed by ELMER SCHUYLER, Brooklyn, New York.

Solve the congruence, $3x^2 + 4x + 5 \equiv 0$, mod. 20 [Reid].

189. Proposed by V. M. SPUNAR, Chicago, Ill.

If p and $p_1 = 2^p - 1$ are primes, then are the numbers of the sequence, $p_1 = 2^p - 1$, $p_2 = 2^{p_1} - 1$, $p_3 = 2^{p_2} - 1$, $\cdots$, $p_n = 2^{p_{n-1}} - 1$ all primes?

SOLUTIONS OF PROBLEMS.

ALGEBRA.

379. Proposed by C. N. SCHMALL, New York, N. Y.

Given $y = ax + b$. If the values of x increase in an arithmetical progression, show that the values of y vary likewise.

SOLUTION BY T. M. BLAKSLEE, Ames, Iowa.

We have

$$y_1 = ax_1 + b, \quad y_2 = ax_2 + b, \quad y_3 = ax_3 + b, \quad \cdots.$$

Hence,

$$y_2 - y_1 = a(x_2 - x_1), \quad y_3 - y_2 = a(x_3 - x_2), \quad \cdots.$$

Since $x_1, x_2, x_3, \cdots, x_n$ are in an arithmetical progression,

$$x_2 - x_1 = x_3 - x_2 = x_4 - x_3, \quad \cdots, \quad = d \text{ say.}$$

Hence,

$$y_2 - y_1 = y_3 - y_2 = y_4 - y_3, \quad \cdots, \quad = ad.$$

Hence,

$$y_2 = y_1 + ad, \quad y_3 = y_2 + ad = y_1 + 2ad,$$

$$y_4 = y_3 + ad = y_1 + 3ad, \quad \cdots, \quad y_n = y_1 + (n-1)ad,$$

an arithmetical progression whose common difference is ad.

Also solved by A. H. Holmes, A. M. Hardin, Elmer Schuyler, W. T. Risley, T. J. Fitzpatrick, and the Proposer.

380. Proposed by J. K. ELWOOD, Lucas, Kansas.

A and B set out to walk around a cinder path a mile in circumference, and walk 3 hours, A walking 8 miles farther than B. Each reduced his rate one mile per hour at the end of the first hour, and again one mile per hour at the end of the second hour, his speed being otherwise uniform. They start in the same direction, but 12 minutes after A has passed B the third time he turns and walks in the other direction until 6 minutes after he has met B the third time, when he returns to his original direction, and overtakes B four times more. Determine their initial velocities.

SOLUTION BY THE PROPOSER.

A gains $2\frac{2}{3}$ miles an hour, or one mile in 22.5 minutes. He overtakes and passes B the first three times in 67.5 minutes, and the last three times in 67.5 minutes. All of A's travel requiring calculation, therefore, occurs in

$$180 - (67.5 + 67.5) = 45 \text{ minutes of the second hour.}$$

Let $x =$ the sum of their rates during second hour.

In the 12 minutes after passing B the third time A gains 8/15 mile. Hence, after A reverses his direction, the three meetings will require them to travel $2\frac{8}{15}$ miles at x miles an hour, or $x/60$ mile per minute. Then $2\frac{8}{15} \div x/60 = 152/x$ is the number of minutes required for the three meetings.

After this they continue in opposite directions 6 minutes, and together travel $x/10$ miles.

Case I.—If they do not meet during the 6 minutes, then A, after resuming his original direction, must first gain $x/10$ miles to overtake B, which requires $x/10 \times 22.5$ minutes. Hence

$$12 + \frac{152}{x} + 6 + \frac{x}{10} \times 22.5 = 180 - (67.5 + 67.5) = 45 \text{ minutes.}$$

or,

$$\frac{9x}{4} + \frac{153}{x} = 27, \tag{1}$$

from which $x = 18 \pm \sqrt{-284}$.

Case II.—If they meet once during the 6 minutes, then A must gain $(x/10 - 1)$ miles, which he will do in $(x/10 - 1) \times 22.5$ minutes.

Then equation (1) becomes

$$\frac{9x}{4} - 22.5 + \frac{152}{x} = 27, \tag{2}$$

from which $x = 18.31$.

Hence A's rate is now 10.49 miles per hour and B's is 7.82 miles per hour. Then A began walking at 11.49 miles per hour and B at 8.82 miles per hour. In this way a complete time-table has been constructed.

GEOMETRY.

407. Proposed by S. LEFSCHETZ, University of Nebraska.

To construct a right triangle knowing the sum of the sides of the right angle and the sum of one of them and the hypothenuse.

I. Solution by A. H. Holmes, Brunswick, Maine.

Let a be the sum of the two sides and b be the sum of one side and the hypothenuse of the required triangle. Draw $AB = 2a$ and extend to C, making $BC = b - a$. On AC describe the semi-circle, ACD. At B erect a perpendicular BD cutting the circumference in D. On DB take $DE = BC$. Then BE is one of the required sides. From O, the middle point of AB, lay off $OF = BE$. Then BF is the other side and BEF is the required triangle. For, letting x and y represent the sides, we have $x + y = a$ and $x + \sqrt{x^2 + y^2} = b$, from which $x = \sqrt{2b(b - a)} - (b - a)$ and $y = b - \sqrt{2b(b - a)}$.

II. SOLUTION BY T. M. BLAKSLEE, Ames, Iowa.

Given $b + a$ and $a + h$, b, a, and h being the base, altitude, and hypothenuse respectively. Draw two parallel rays, l and s at distance $a + h$. On l lay off $AD = b + a$. Draw DE making $\angle ADE = 45°$. The ray of DE contains the vertex B of the required triangle ACB. This vertex is also on the parabola having s as directrix and A as focus.

Having found B, draw BC perpendicular to l and join A and B.

Excellent solutions were also received from A. M. Harding, C. N. Schmall, H. C. Feemster, Levi S. Shively, Elmer Schuyler, W. T. Risley, and Francis C. Rust.

408. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given a point A on a circle and a chord of the circle; to draw a chord through A so that it shall be bisected by the given chord.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let C be the center of the circle. On CA as a diameter describe a circle. This circle will cut the chord, in general, in two points P_1, P_2. Join A to P_1 and A to P_2. Then these are the required chords. If A be on the minor arc of the circle there will be two solutions. If A be on the major arc there will be two solutions, one solution, or no solution according as d is less than, equal to, or greater than $r - c$, where d is the distance from A to the given chord, c is distance from center to the given chord and r is the radius of given circle.

Also solved by T. M. Blakslee, C. N. Schmall, A. H. Holmes, Levi S. Shively, Francis C. Rust, G. W. Hartwell, W. R. Lebold, H. C. Feemster, and Elmer Schuyler.

CALCULUS.

326. Proposed by C. N. SCHMALL, New York City.

Prove that

$$\int_0^\infty \frac{(\tan^{-1} ax)^2 - (\tan^{-1} bx)^2}{x}\, dx = \frac{\pi^2}{4}(\log a - \log b).$$

SOLUTION BY THE PROPOSER.

Take the integral,

$$u = \int_0^\infty [f(x) - f(kx)]\frac{dx}{x}. \tag{1}$$

Substitute for x the successive values kx, k^2x, k^3x, $\cdots$, k^nx.

We then have

$$u = \int_0^\infty [f(kx) - f(k^2 x)]\frac{dx}{x}, \tag{2}$$

$$u = \int_0^\infty [f(k^2 x) - f(k^3 x)]\frac{dx}{x}, \tag{3}$$

$$\cdots\cdots\cdots\cdots$$

$$u = \int_0^\infty [f(k^{n-1} x) - f(k^n x)]\frac{dx}{x}. \tag{n}$$

Adding these equations, we get

$$nu = \int_0^\infty [f(x) - f(k^n x)] \frac{dx}{x} = \int_0^\infty [f(x) - f(rx)] \frac{dx}{x},$$

where $r = k^n$.

Now, suppose n to become infinite, r remaining constant, then k will approach unity as a limit, and we have

$$\int_0^\infty [f(x) - f(rx)] \frac{dx}{x} = n \int_0^\infty [f(x) - f(r^{1/n} x)] \frac{dx}{x}.$$

The right member of this equation (in the limit) reduces to the indeterminate form 0/0. Differentiating in this form with respect to $1/n$, we get

$$-\int_0^\infty f'(x) \log r dx = \log r[f(0) - f(\infty)].$$

Now in the given integral, $f(x) = (\tan^{-1} ax)^2$. Hence,

$$f(0) = 0, \qquad f(\infty) = \tfrac{1}{4}\pi^2;$$

also,

$$r = \frac{b}{a}.$$

Hence the integral is equal to $\frac{1}{4}\pi^2 (\log a - \log b)$.

This problem may be readily solved by Frullani's Theorem, Williamson's *Integral Calculus*, page 155. Professor J. Scheffer solved the problem by means of this Theorem, but failed to notice that in the above problem $\phi(ax) = [\tan^{-1} ax]^2$ instead of $\tan^{-1} ax$. Ed. F.

327. Proposed by RICHARD P. LOCHNER, Philadelphia, Pa.

A hound is at the middle point of the side of a square field and a fox is at an adjacent corner. How far will the hound run to catch the fox if the fox runs on the perimeter of the field and the hound runs directly towards the fox at all times, the hound running n times as fast as the fox. Where will the race end?

No solution of this problem has been received.

328. Proposed by M. E. GRABER, Tiffin, Ohio.

Prove that

$$\frac{E_m}{\pi/2m}\int_0^{\pi/2m}\left[\sin a + \sin\left(\frac{\pi}{m} + a\right) + \cdots + \sin\left(\pi\frac{m-1}{m} + a\right) da\right] = \frac{2mE_m}{\pi}.$$

SOLUTION BY A. M. HARDING, University of Arkansas.

From trigonometry we have

$$\sin a + \sin\left(\frac{\pi}{m} + a\right) + \cdots + \sin\left(\pi\frac{m-1}{m} + a\right) = \frac{\sin\left(a + \dfrac{m-1}{2m}\cdot\pi\right)}{\sin\dfrac{\pi}{2m}}.$$

Hence the given integral takes the form

$$\frac{E_m}{\frac{\pi}{2m}} \cdot \frac{1}{\sin\frac{\pi}{2m}} \int_0^{\pi/2m} \sin\left(a + \frac{m-1}{2m} \cdot \pi\right) da$$

$$= \frac{2mE_m}{\pi} \cdot \frac{1}{\sin\frac{\pi}{2m}} \cdot \left[-\cos\left(a + \frac{m-1}{2m}\pi\right)\right]_0^{\pi/2m}$$

$$= \frac{2mE_m}{\pi} \cdot \frac{1}{\sin\frac{\pi}{2m}} \cdot \sin\frac{\pi}{2m} = \frac{2mE_m}{\pi}.$$

Solved similarly by Geo. W. Hartwell and J. Scheffer.

329. Proposed by C. N. SCHMALL, New York City.

Show that the general differential equation

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P^2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_{n-1}\frac{dy}{dx} + P_ny = V,$$

where P_1, P_2, $\cdots$ P_n, V are known functions of x, has a solution in the form

$$y - v_1\int v_2dx \int v_3dx \cdots \int \frac{Vdx}{v_1v_2v_3 \cdots v_n}.$$

SOLUTION BY A. M. HARDING, University of Arkansas.

Let $y = v_1z_1$. Then the given differential equation becomes

$$v_1\frac{d^nz_1}{dx^n} + Q_1\frac{d^{n-1}z_1}{dx^{n-1}} + \cdots + Q_{n-1}\frac{dz_1}{dx}$$
$$+ z_1\left[\frac{d^nv_1}{dx^n} + P_1\frac{d^{n-1}v_1}{dx^{n-1}} + \cdots + P_{n-1}\frac{dv_1}{dx} + P_nv_1\right] = V. \quad (1)$$

Now suppose the expression in brackets, when equated to zero, is satisfied by v_1. In that case this expression will vanish, and we have left a linear equation of the $(n-1)$th degree in dz_1/dx.

Putting then

$$\frac{dz_1}{dx} = y_1, \quad (2)$$

and $Q_r = v_1R_r$, we have

$$\frac{d^{n-1}y_1}{dx^{n-1}} + R_1\frac{d^{n-2}y_1}{dx^{n-2}} + \cdots + R_{n-1}y_1 = \frac{V}{v_1}.$$

Now this equation has the same form as the original equation and is one degree lower. Proceeding in the same way we at last get

$$y_n = \frac{V}{v_1v_2v_3 \cdots v_{n-1}v_n},$$

but

$$y = v_1 z_1 = v_1 \int y_1 dx,$$

by (2). Similarly,

$$y_1 = v_2 z_2 = v_2 \int y_2 dx,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$y_{n-1} = v_n z_n = v_n \int y_n dx.$$

Hence

$$y = v_1 \int v_2 dx \int v_3 dx \cdots \int \frac{V dx}{v_1 v_2 v_3 \cdots v_{n-1} v_n}.$$

Also solved in the same manner by the Proposer.

MECHANICS.

263. Proposed by C. N. SCHMALL, New York City.

A railroad car is rounding a curve of radius r with a velocity v, $2d$ being the distance between the rails. If h is the distance of its center of gravity above the rails, g having the usual meaning, show that the weight of the car is divided between the outer and inner rails in the ratio $\dfrac{dgr + v^2h}{dgr - v^2h}$.

I. SOLUTION BY H. PRIME, Boston, Mass.

Let R and R' be the reaction of the weight on the outer and inner rails respectively. Then we have for the moments

$$\text{about inner rail,} \qquad 2dR = dmg + \frac{hmv^2}{r};$$

and

$$\text{about outer rail,} \qquad 2dR' = dmg - \frac{hmv^2}{r}.$$

Whence

$$\frac{R}{R'} = \frac{dgr + v^2h}{dgr - v^2h}.$$

II. SOLUTION BY J. SCHEFFER, Hagerstown, Md.

Denoting the pressures upon the rails by P and Q, and the angle which a plumb-line makes with the vertical by ϕ, the vertical strikes the distance between the rails at a point distant $d - h\tan\phi$ and $d + h\tan\phi$ from the rails respectively. Hence $P(d - h\tan\phi) = Q(d + h\tan\phi)$. Then

$$\frac{P}{Q} = \frac{d + h\tan\phi}{d - h\tan\phi}.$$

But $\tan\phi = v^2/rg$, and

$$\frac{P}{Q} = \frac{dgr + hv^2}{dgr - hv^2}.$$

No solution of 264 has been received. We shall be pleased to receive solutions of unsolved problems as well as a variety of good problems in the several problem departments.

NEWS AND NOTES.

FLORIAN CAJORI, CHAIRMAN OF THE COMMITTEE.

New subscribers to the MONTHLY are invited to read the reflections of Editor Finkel, in the December issue, on the nineteen years' history of this journal.

The spring meeting of the Chicago Section of the American Mathematical Society will be held at the University of Chicago, on Friday and Saturday March 21, 22.

At the annual meeting of the American Mathematical Society in Cleveland, Professor E. B. VAN VLECK, of the University of Wisconsin, was elected president. The next summer meeting of the Society, including a colloquium, will be held at Madison, Wis., early in September, 1913.

We notice with regret the death of Professor WILHELM FIEDLER of the Polytechnicum in Zürich, known for his work in geometry and his translations into German of the books of George Salmon.

In this country have occurred the deaths of OLIVER CLINTON WENDELL, professor of astronomy at Harvard University, of EBEN LOOMIS who was for half a century connected with the Nautical Almanac Office, and of EDWIN SMITH who was connected with the U. S. Coast and Geodetic Survey since 1870.

The first part of EULER'S *Commentationes* on elliptic integrals has just appeared as Vol. XX of the first series of Euler's *Opera Omnia.*

Professor JACQUES HADAMARD has been elected a member of the Paris Academy of Sciences in the section of geometry, in succession to the late HENRI POINCARÉ.

On January 19, 1913, occurred in Denver the death of Robert Gauss, who had been long connected with the *Denver Republican.* On the same day died also his brother, Charles H. Gauss, of St. Louis. They were sons of Eugene Gauss, and grandsons of the mathematician, CARL FRIEDRICH GAUSS.

Announcement was made of the death, on December 7, of SIR GEORGE HOWARD DARWIN, Plumian professor of astronomy and experimental philosophy at the University of Cambridge. He was a leading authority on tidal phenomena. He was president of the fifth international congress of mathematicians which was held in Cambridge University, August 21 to 28, 1912. In his address before the Congress he spoke of the Cambridge School of applied mathematics and its work during the last century.

"The Cambridge School of Mathematics" is the title of an historical article by W. W. ROUSE BALL in the *Mathematical Gazette* for July, 1912. Without entering much into critical and philosophical inquiries, Ball tells in a pleasant way of mathematical research and teaching at Cambridge from the time of Tonstall, Recorde and John Dee to the present time. This number of the *Gazette* is

adorned by fine page portraits of Sir Isaac Newton, Sir George Darwin, Sir Joseph Larmor, Professor E. W. Hobson and Professor A. E. H. Love.

The collected papers of SOPHUS LIE are to be published by the firm of B. G. Teubner in seven volumes, under the editorship of Friedrich Engel. Subscription prior to April 1, 1913, is at the reduced rate of about 160 marks. The first two volumes will be on geometry, the next two on differential equations, the two following on transformation groups, while the last volume will be reserved for the publication of residual manuscripts.

At the Cleveland meeting of the American Association for the Advancement of Science held last December, Professor F. R. MOULTON, of the department of astronomy at the University of Chicago, was elected secretary of Section A, to succeed Professor G. A. Miller, of the University of Illinois, who had held this position since 1907.

After this year, mathematics is to be an elective subject in the College of Science, Literature, and Arts at the University of Minnesota. For the last five years, college algebra and trigonometry have been required of those students who entered the college without having had both higher algebra and solid geometry. Before that time, one year of work in college mathematics was required of all candidates for the Bachelor of Arts degree.

A most interesting account of HENRI POINCARÉ and his work, from the pen of Professor F. R. Moulton, appeared in the December number of *Popular Astronomy*. Another sketch which touches a different phase of the work of Poincaré appeared in *Nature* (London) of November 28, 1912.

The University of Oregon has begun the publication of a monthly, called the *Extension Monitor*. In the first number (January, 1913) Professor E. E. DE COU gives a brief account of the work of the international commission for the improvement of mathematical teaching.

Professor W. E. BYERLY, of Harvard University, will become professor emeritus at the close of the present academic year. He is well known for his texts on the calculus and on Fourier's series and spherical harmonics, as well as for his unusual ability as a lecturer and teacher.

Isis is the name of a new international journal to be devoted to the history of science, under the editorship of GEORGE SARTON, D.Sc., of Wondelgem-lez-Gand in Belgium. On the "Comité de patronage" are 30 names, including those of two Americans, JACQUES LOEB, of the Rockefeller Institute for Medical Research, and DAVID EUGENE SMITH, of Columbia University.

The Royal Society of Edinburgh is preparing to celebrate the tercentenary of the first important British contribution to mathematical science, namely the invention of logarithms by NAPIER of Merchiston in 1614. What form the cele-

bration will take is not yet announced, but probably a bird's eye view of mathematical progress along the lines of algebra will be one of the features.

In the students' section of the mathematics club of the University of Illinois the monthly programs for the present academic year include the following papers: Errors in mathematical literature, by Professor G. A. Miller; Construction of regular polygons, by E. A. Kircher; The practical use of the slide rule, by C. R. Horrell; Mathematical induction, by C. C. Yen; The Hindu-Arabic numerals, by F. A. Murray; The calculation of π, by Miss Peach Andrews; The theory of the gyroscope, by Miss Mildred Seyster; Mathematical models, by Professor A. Emch.

Professor Favaro, of Padua, at the October meeting of the Italian society for the advancement of science, gave an account of an unpublished translation of the writings of ARCHIMEDES, found among the Galileoan manuscripts in the Library of Florence. The translation was made by Vincenzio Viviani, a pupil of Galileo, and was apparently intended for publication.

The courses in mathematics to be offered at the University of Chicago in the summer session, 1913, include the following: By Professor E. H. MOORE, Seminar on point set theory, Fourier's series, Trigonometry; by Professor OSKAR BOLZA, Linear integral equations, Theory of functions of a complex variable; by Professor F. R. MOULTON, Theory of differential equations, Descriptive astronomy; By Professor H. E. SLAUGHT, Differential equations, Integral calculus; by Professor J. W. A. YOUNG, Differential calculus, Critical review of secondary mathematics; by Professor E. J. WILCZYNSKI, Theory of equations, Analytic geometry; by Professor G. A. BLISS, Projective geometry, College algebra; by Professor A. C. LUNN, Applied mathematics, Graphical analysis; by Professor W. D. MACMILLAN, Celestial mechanics, Observational astronomy.

The United States Bureau of Education has just published a *Bibliography of the Teaching of Mathematics* covering the period from 1900 to 1912, by David Eugene Smith and Charles Goldziher. This Bulletin gives 1,849 titles of books and articles on the teaching of mathematics that have appeared since 1900. The Bulletin will be sent gratis upon application to the United States Commissioner of Education, Washington, D. C.

Some of our readers may not agree with all the conclusions reached by the author of "Minimum Courses in Engineering Mathematics" in this issue. It is needless to say that the Editors assume no responsibility for the views expressed in any signed article. They will welcome the expression of dissenting views in the interest of helpful discussion along this line.

Since the last issue, Northwestern University has been added to the list of institutions contributing to the subsidy fund of the MONTHLY, making the total number now ten universities and two colleges. Professor D. R. CURTISS becomes a member of the Editorial Board whose assistance will be greatly valued.

We are again unable to issue the MONTHLY on time, and it may require still another month for the printer to catch up with the schedule. This number is also larger than normal, though we shall be pleased if subscription returns may make it possible to continue this larger size.

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HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

III. THE CREATION OF A THEORY OF LOGARITHMS OF COMPLEX NUMBERS BY EULER.

1747–1749.

It is desirable to look back, for a moment, over the 35 years of history of logarithms of negative and complex numbers. Thus far only three mathematicians have attempted to unravel the mysteries of this subject, namely Leibniz, John Bernoulli I and Euler. Their discussions were carried on entirely by letter; these letters were not published at the time. No articles or memoirs on this controversy had reached the press. The question had not been brought to the attention of the mathematical public.

Publicity on this subject began in 1745. In that year was published the correspondence between Leibniz and John Bernoulli I.[1] The reading of that correspondence acted as a tremendous stimulus upon Euler. As a boy of 20 he himself, as we have seen, had corresponded on this subject with his revered master, John Bernoulli I. That correspondence had set bare serious difficulties of the subject, but had not removed them. Since that time he had discovered the exponential expressions for sin x, cos x, and cos $x + i$ sin x; he had acquired a deeper insight into the properties of imaginary numbers. It was in 1745 that he completed his manuscript on the *Introductio,* which was issued from the press three years later. Two years later (in 1747) he carried on his classic researches on logarithmic theory. He discussed this subject in his correspondence with D'Alembert. Some of Euler's letters on this subject have been published only recently. For that reason not even Lampe's historical essay is based on a study of all the material that is now available. For convenience of reference we give Euler's writings on this subject, the year in which they were penned, and the year of publication. The first two publications we have already reviewed.

[1] *Virorum celeberr. Got. Gul. Leibnitii et Johan. Bernoullii commercium philosophicum et mathematicum. Lausannæ et Genevæ,* 1745.

	Time of Writing.	Time of Publication.
1. Correspondence between Euler and John Bernoulli I	1727–29	1902
2. Euler's *Introductio in analysin*	1745	1748
3. Euler's letters to D'Alembert	1747–8	1886[1] 1907[2]
4. Euler's article *Sur les logarithmes*	1747	1862[3]
5. Euler's letters to D'Alembert, dated Dec. 27, 1748; Jan. 3, 1750; July 26, 1763; Dec. 20, 1763		1768[4]
6. Euler's letter to D'Alembert . . . Feb. 15,	1748	1911[5]
7. Euler's article *De la controverse entre* MRS. LEIBNITZ ET BERNOULLI	1749	1751[6]
8. Euler's article *Recherches sur les racines imaginaires*	1749	1751[7]

Euler's letter of April 15, 1747, to D'Alembert, written in French, is seen from the context to be in response to an argument advanced by D'Alembert in letters now lost. The correspondence between the two men seems at times to have been quite lively. We give the synopsis of five letters written by Euler from Berlin.

April 15, 1747. Euler to D'Alembert: I must oppose your argument that e^1 can have a positive and a negative value. I admit the value to be arbitrary, say 10 or 2.718 . . . , but as soon as e in $y = e^x$ is assigned a definite value, the system of logarithms of all numbers is fixed, as is also the curve $y = e^x$. One cannot assign e two different values at the same time, without making the resulting curve the composite of two distinct curves. If $e = 1 + 1 + 1/(1 \cdot 2) + 1/(1 \cdot 2 \cdot 3) + \cdots$ then it is clear that the logarithms of negative numbers must be "impossible," for no value of x can be found which makes e^x or $1 + x/1 + xx/1 \cdot 2 + \cdots$ negative. To you it seems paradoxical that $ly = l(-y)$, but for *any* value of a we have $d(ly) = d(lay)$, hence also for $a = -1$. The argument by which you prove that $l(-1) = 0$ enables you to prove that $l\sqrt{-1} = 0$; for since $\sqrt{-1} \cdot \sqrt{-1} = -1$, you have $l\sqrt{-1} + l\sqrt{-1} = l(-1) = 2l\sqrt{-1} = \frac{1}{2} \log (+1)$ and $l\sqrt{-1} = \frac{1}{4}l(+1) = 0$. You will admit that the logarithms of imaginary numbers are not real; else $(l\sqrt{-1})/\sqrt{-1}$ could not express the quadrature of the circle. Let $(l\sqrt{-1})/\sqrt{-1} = \alpha$, then $l\sqrt{-1} = \alpha\sqrt{-1}$, which is imaginary. Now, if $l\sqrt{-1}$ is imaginary, why should not $2l\sqrt{-1} = l(-1)$ be so? Similarly it would follow that the logarithm of an imaginary cube root of 1 would be 0, as are also $l(+1)$, $l(-1)$, $l\sqrt{-1}$, etc. This is untenable. You will reply that even $l(+1)$ must be imaginary, for $l(+1) = 2l(-1) = 4l\sqrt{-1} = 3l\,(-1 + \sqrt{-3})/2 = \cdots$. This is exactly what I claim, for $l(+1)$ has an infinite number of distinct values, of which all are imaginary except 0. Let $l(+1) = 0, \alpha, \beta, \gamma, \delta, \epsilon, \cdots$, then $\frac{1}{2}\alpha, \frac{1}{2}\gamma, \frac{1}{2}\epsilon, \cdots$ are the logarithms of -1, all imaginary; not every half of the values of $l(+1)$ is a value of $l(-1)$; for, -1 is only one value of $\sqrt{+1}$. The other value $+1$ has the logarithms $\frac{1}{2} \cdot 0, \frac{1}{2}\beta, \frac{1}{2}\delta, \cdots$, which are same as $0, \alpha, \beta, \gamma, \delta, \epsilon, \cdots$; for we have $\frac{1}{2}\beta = \alpha$, $\frac{1}{2}\delta = \beta, \cdots$. Similarly for the logarithms of the cube roots of 1. I can give the actual values. Letting π be the circumference (!) of a circle of unit radius, the values of $\log(+1)$ are: $0, \pm \pi\sqrt{-1}, \pm 2\pi\sqrt{-1}, \pm 3\pi\sqrt{-1}, \cdots$. Those of $l(-1)$ are: $\pm \frac{1}{2}\pi\sqrt{-1}, \pm \frac{3}{2}\pi\sqrt{-1}, \pm \frac{5}{2}\pi\sqrt{-1}, \cdots$. Generally: $l(1^p) = \pi(mp + n)\sqrt{-1}$, $l(-1)^p = \pi(\frac{1}{2}p + mp + n)\sqrt{-1}$, where m and n are $+$ or $-$ integers. Here all difficulties disappear which arise in the effort to make the logarithms of negative numbers real. According to the formula $2l(-1) = l(+1) = 0$ we would have $l\sqrt{-1} = 0$ and $l\frac{1}{2}(-1 + \sqrt{-3}) = 0$. You say that in $e^x = y$, y may be

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[1] *Bullettino* BONCOMPAGNI, Vol. 19, 1886, pp. 136–148.

[2] E. LAMPE in *Festschr. z. Feier d. 200. Geburtstages* L. EULERS, Leipzig und Berlin, 1907, pp. 125–135.

[3] EULER, *Opera posthuma*, 1 (1862), pp. 269–281.

[4] *D'Alembert, Opuscules mathématiques*, Vol. IV, Paris, 1768, pp. 342–343, 146, 162.

[5] *Bibliotheca mathematica*, 3d S., Vol. 11, 1911, pp. 220–226.

[6] *Mém. de l'acad. de sc. de Berlin*, Vol. 5 (1749), 1751, pp. 139–179.

[7] Same volume, pp. 270–288.

$+$ and also $-$, when $x = \frac{1}{2}$. But since e^x represents the value of the series $1 + x + x^2/(1 \cdot 2) + \cdots$, I rest on solid ground when I say that e^x stands for only one value, which is positive even when x is fractional.

The logarithms of negative and imaginary numbers had been under discussion, off and on, by a few mathematicians for over 35 years, but the above letter of Euler to D'Alembert is the first investigation which really penetrates the subject and yields substantial results. The theorem was announced therein that Log n has an infinite number of logarithms which are all imaginary, except when n is a positive number, in which case one logarithm out of this infinite number is real. It is also interesting to observe that Euler gives in this letter a new definition of the exponential e^x. He lets e^x represent the value of the exponential series, $1 + x + x^2/(1 \cdot 2) + \cdots$, and thereby he takes a point of view which has since found adoption in the theory of functions. D'Alembert's reply to Euler's letter is not known. Since Euler's main results were stated in the above letter without proof, it is not surprising if D'Alembert was not convinced. Some idea of D'Alembert's position may be gathered from Euler's next letter.

Aug. 19, 1747. Euler to D'Alembert: In your article on integrals, recently published in the second volume of our memoirs, I have crossed out the paragraph on log (-1), as you directed. Your statement, that $l(-x)$ can be expanded into a series whose value is real, is incomprehensible to me. When you say that e must be considered, not a parameter of the logarithmic curve, but the ordinate when $x = 1$, that, therefore, e may be $+$ as well as $-$, then I have as much right to claim that the logarithmic curve has not only two branches, but any number. If $x = l(+y)$ and $x = l(-y)$ yield two branches, then $x = l(my)$, the differential equation of which is the same for all values of m, yields a branch for every possible value of m. In e^x you take $x = k/g$, where $k : g$ = odd number : even number. With equal right we can take $k : g$ = even number : odd number, or = odd number : odd number, and thereby reach conclusions at variance with yours. Your arguments fail to establish the formula $l(+x) = l(-x)$. As regards $(l\sqrt{-1})/\sqrt{-1}$, I maintain that it can have no other values than $\frac{1}{2}(4n + 1)\pi$, where n is any integer, π the circumference of a circle of unit diameter; hence this formula can never yield 0. I have sent to the Academy a memoir, which appears to me to remove all difficulties on this subject which formerly perplexed me greatly.

P. S. You admit that $l(+1) = \pm 2n\pi\sqrt{-1}$ and $l(-1) = \pm (2n - 1)\pi\sqrt{-1}$, but you claim that 0 is among the logarithms of -1. Since twice the logarithm of -1 is $l(+1)$, we would have $l(+1) = \pm 2n\pi\sqrt{-1}$ and also $= \pm (2n - 1)\pi\sqrt{-1}$. You admit that $l\sqrt{-1} = \pm \frac{1}{2}(4n \pm 1)\pi\sqrt{-1}$, and $l(-\sqrt{-1}) = \pm \frac{1}{2}(4n \pm 1)\pi\sqrt{-1}$, but assert that these formulæ do not yield all the logarithms, that they omit the logarithm 0. If so, then $l(+1)$, or $l(+\sqrt{-1}) + l(-\sqrt{-1})$, may equal $\pm \frac{1}{2}(4n \pm 1)\pi\sqrt{-1}$. You claim that 0 may be the logarithm of the higher imaginary roots of unity; hence $l(+1)$ would be represented by $a\sqrt{-1}$, whatever number a may be. Consequently, $l(+1)$ would be wholly indeterminate. This conclusion overthrows your contentions. My results are in perfect harmony with each other and yield no such indeterminateness.

Dec. 30, 1747. Euler to D'Alembert: I learn through Mr. de Maupertius that you are suspending mathematical research, in order to recuperate from ill-health. I shall therefore not trouble you with matters about imaginary logarithms. I have hardly anything to add to what I said before. I doubt whether my paper on this subject will remove all the doubts which you have brought.

Feb. 15, 1748. Euler to D'Alembert: The equation $y = 2^x$ gives a continuous curve above the axis of x, but if $x = \frac{1}{2}$, then $y = +\sqrt{2}$ and $-\sqrt{2}$, and I grant that there is a conjugate point below the axis of x. Taking $x = n/2$, there is an infinity of such points below the x-axis, which are isolated from each other.[1] The equation $y = (-2)^x$ gives an infinity of

[1] " . . . je pretend que chacun de ces points est isolé sans liaisons avec les voisins, quoique leurs distances soient meme infiniment petites."

isolated points and no continuous curve whatever. $y = e^x$ represents a continuous curve above the x-axis and isolated points below.

It is seen that Euler recedes here from his definition of e^x as the sum of the series $1 + x + x^2/(1 \cdot 2) + \cdots$, adopted in his letter of April 15, 1747. He assigns no reason for his change of view-point, nor does he explain how this admission affects the question under controversy.

Sept. 28, 1748. Euler to D'Alembert: The subject of imaginary logarithms is no longer so clearly in my mind that I could reply rigorously to your recent remarks. I must wait till I can examine this matter again.

The article which Euler reports in his letter of August 19, 1747, as having been sent by him to the Berlin Academy, is without doubt the article[1] which was first published in 1862 and entitled *Sur les logarithmes des nombres négatifs et imaginaires.*[2] Euler begins by referring to the "grande controverse" between Leibniz and John Bernoulli I which ended in disagreement between these great men who were in perfect harmony on all other parts of analysis. The glory of infallibility of this science receives a severe blow, if it presents questions in which it is impossible to ascertain the truth and end all dispute. It is my hope by the present research to settle the controversy on the logarithms of negative numbers. Reviewing the arguments of Leibniz and John Bernoulli I, Euler remarks that Leibniz's demur to B.'s argument—that $l(+x) = l(-x)$ since $dlx = dl(-x) = dx/x$, on the ground that the rule for finding lx holds only when x is positive—shakes the very foundation of analysis, whose rules of operation are taken to be general and applicable to quantities of all kinds. B.'s error lies in claiming $lx = l(-x)$ on the ground that $dlx = dl(-x)$; the same argument yields the absurdity $lnx = lx$. B.'s claim, that the logarithmic curve is double, involves the argument just mentioned; if $lx = l(-x)$ then similarly $lnx = lx$, and the curve has an infinite number of branches. A differential equation always yields on integration a family of curves. B. and others claim that the asymptote to the logarithmic curve is a diameter of it, saying that $dx = dy/y^n$ yields a curve which in general has a diameter when n is odd and has one therefore when $n = 1$. Euler explains this point and closes with the novel statement that the question of the doubleness of the logarithmic curve is not necessarily connected with the question whether $l(-x)$ is real or imaginary. Euler then defines a logarithm; if $x = ly$, then $y = e^x$, e being a constant. "Donc le logarithme x d'un nombre proposé $= y$ n'est autre chose que l'exposant de la puissance de e qui est égale au nombre y" (p. 273). In hyperbolic logarithms, if ω is infinitesimally small, then $l(1 + \omega) = \omega$ and $e = 2{,}718 \cdots$. If y is negative, no real value of x satisfies $y = e^x$. To be sure, when $x = \frac{1}{2}$, $y = \pm \sqrt{e}$; but when $x = 2$, it is not true that $y = \pm ee$ and that $x = l \pm ee$. Hence, the statement that the logarithms of negative numbers are real, is certainly not a general truth. "Mais pour ce qui regarde l'ambiguité de la formule e^x, dans le cas oú x est une fraction d'un dénominateur pair, je ne sais pas si on la peut admettre dans les logarithmes.

[1] See G. Eneström, "Verzeichniss d. Schriften L. Eulers," Leipzig, 1910, *Jahresber. d. deutschen Math. Vereinig., Ergänzsb., IV Bd., 1 Lief.*, pp. 42, 202, Nos. 168, 807.

[2] L. Euleri *Opera posthuma*, Tomus prior, Petropoli, 1862, pp. 269–281.

Car, ayant égard à la nature et à l'usage des logarithmes, il semble qu'à chaque logarithme ne puisse répondre qu'un seul nombre" (p. 274). If some one says that $e^0 = e^{0/2} = \sqrt{e^0} = \sqrt{1} = \pm 1$, then by the same argument $x^1 = x^{2/2} = \pm x$, "et de plus que $a + x$ serait la même chose que $a - x$," whence the absurdity "toutes les quantités sont égales entre elles." But if $l(-1)$ is not 0, it must be imaginary, and $l(-y) = l(-1) + ly$ must be imaginary. That x can be the logarithm of only one number y is confirmed by the series $e^x = 1 + x + x^2/(1\cdot 2) + \cdots$. If one insists that, for $x = \frac{1}{2}$, $e^x = \pm \sqrt{e}$ it follows that the logarithms of imaginary numbers are real; for, when $x = \frac{1}{3}$, $\frac{1}{3}$ is the logarithm of the real cube root of e, as well as of the two imaginary ones. But B. has made the beautiful discovery that $\pi = 2(l\sqrt{-1})/\sqrt{-1}$; it follows that $l\sqrt{-1}$ must be imaginary and that $l\sqrt{-1} = 0$ cannot be true. But if we let $l(-1) = p$, p imaginary, we encounter $l(-1)^2 = l(+1) = 2p = 0$, which is contrary to hypothesis and leads up to $l\sqrt{-1} = 0$, as before. "Voilà donc des contradictions assez palpables qu'on recontre, de quelque côté qu'on se tourne; . . . ce serait sans doute une tache indélébile dans l'analyse, si la doctrine des logarithmes était tellement remplie de contradictions, qu'il fût impossible de trouver une conciliation" (p. 275). Euler tries the experiment of letting the logarithms of the three cube roots of y be different from each other, namely equal to $x/3$, $x'/3$ and $x''/3$. Of these three logarithms the first is real, the other two imaginary; but the triple of each must be x. "Cette explication me paraissait bien extrêmement paradoxe et insoutenable, mais pourtant moins absurde que les contradictions que j'aurais été obligé d'admettre dans la théorie des logarithmes des nombres négatifs et imaginaires" (p. 276). But the fertile mind of Euler had not exhausted all the possibilities; he comes out with the pregnant remark: "Après avoir bien pesé toutes les difficultés que je viens d'étaler, j'ai trouvé qu'elles ne viennent que de ce que nous supposons que chaque nombre n'a qu'un seul logarithme" (p. 276). Admitting that a number may have many, in fact an infinite number, of logarithms, all difficulties vanish. To show that this number is really infinite, Euler takes the circle as better suited for the study of logarithms than the logarithmic curve; he develops by the aid of the integral calculus the equation $\sqrt{-1}\varphi = l(\cos\varphi + \sqrt{-1}\sin\varphi)$, where he writes for φ the more general value $\varphi \pm 2n\pi$, n being any integer. From this formula he derives the logarithms of 1, -1, $-a$, $\sqrt{-1}$, and remarks that Leibniz was correct in claiming that the logarithms of negative numbers are imaginary. He tests his results on products, quotients and powers of positive, negative and imaginary numbers, and remarks, "l'on trouvera constamment un merveilleux accord avec la vérité" (p. 280). Using De Moivre's theorem he obtains finally

$$l1^{\mu/\nu} = \frac{1}{\nu}(\pm 2\mu m \pm 2\nu n)\pi\sqrt{-1}$$

and

$$l(-1)^{\mu/\nu} = \frac{1}{\nu}(\mu \pm 2\mu m \pm 2\nu n)\pi\sqrt{-1},$$

where m and n are any integers.

assumption that a number had only one logarithm. He tried the assumption that there was an infinite number of them, and this worked! How easy all this is to a modern reader, but how hard it was to Euler. His labors in connection with this great research remind us of Kepler's struggle with the orbit of Mars. There is a difference, to be sure. Kepler had to construct a theory which would conform with the facts of nature as revealed by Tycho Brahe's telescope; Euler had to construct a theory which would conform with the demands for consistency in logic. Aside from this the procedure was the same. Kepler made several successive guesses as to what the orbit of Mars might be, only to find that he guessed wrong. Finally it occurred to him to try an *ellipse;* at last he found that he had guessed correctly. The reader of Euler's works will find that in other mathematical topics as well, particularly in the theory of numbers, Euler often followed methods closely akin to those of the student of natural science. And yet, Sir William Hamilton and Thomas Huxley would have us believe that mathematics is a science which knows nothing of observation and experiment!

Euler's paper of 1749 bears the title, *De la controverse entre Mrs. Leibnitz et Bernoulli sur les logarithmes des nombres negatifs et imaginaires.*[1] It starts out with a critical historical account of the controversy. The strictly constructive part of the article consists of a theorem and four problems. The theorem is that *there is an infinity of logarithms for every number.* The proof of this theorem was based in the 1747 article upon the relation $i\varphi = \log(\cos\varphi + i\sin\varphi)$. Here in the 1749 article it is based upon the assumption that $l(1 + \omega) = \omega$, ω being infinitely small. Presumably this relation was based on the series $\log(1 + x) = x - x^2/2 + x^3/3 - \cdots$. Euler's proof of 1749 plays so large a part in logarithmic theory during the half century following, that it should be reproduced here in full. As will be seen later, Euler's reasoning failed to carry conviction. The proof is as follows:

Je me bornerai ici aux logarithmes hyperboliques, puisqu'on sait que les logarithmes de toutes les autres espèces sont à ceux-cy dans un rapport constant, ainsi quand le logarithme hyperbolique du nombre x est nommé $= y$, le logarithme tabulaire de ce même nombre sera $= 0,4342944819\cdot y$. Or le fondement des logarithmes hyperboliques est, que si ω signifie un nombre infiniment petit, le logarithme du nombre $1 + \omega$ sera $= \omega$, ou que $l(1 + \omega) = \omega$. De là il s'ensuit que $l(1 + \omega)^2 = 2\omega$; $l(1 + \omega)^3 = 3\omega$, & en général $l(1 + \omega)^n = n\omega$. Mais puisque ω est un nombre infiniment petit, il est évident, que le nombre $(1 + \omega)^n$ ne sauroit devenir égal à quelque nombre proposé x, à moins que l'exposant n ne soit un nombre infini. Soit donc n un nombre infiniment grand, & qu'on pose $x = (1 + \omega)^n$, & le logarithme de x, qui a été nommé $= y$, sera $y = n\omega$. Donc pour exprimer y par x, la première formule donnant $1 + \omega = x^{1/n}$ & $\omega = x^{1/n} - 1$, cette valeur étant substituée pour ω dans l'autre formule produira $y = nx^{1/n} - n = lx$. D'où il est clair que la valeur de la formule $nx^{1/n} - n$ approchera d'autant plus du logarithme de x, plus le nombre n sera pris grand; & si l'on met pour n un nombre infini, cette formule donnera la vraye valeur du logarithme de x. Or comme il est certain, que $x^{\frac{1}{2}}$ a deux valeurs différentes, $x^{\frac{1}{3}}$ trois, $x^{\frac{1}{4}}$ quatre, & ainsi de suite, il sera également certain, que $x^{1/n}$ doit avoir une infinité de valeurs différentes, puisque n est un nombre infini. Par conséquent cette infinité de valeurs différentes de $x^{1/n}$ produira aussi une infinité de valeurs différentes pour lx, de sorte que le nombre x doit avoir une infinité de logarithmes. C. Q. F. D.

Letting in $y = nx^{1/n} - n = lx$, $x = 1$, he gets $(1 + y/n)^n - 1 = 0$. The factors of any binomial $p^n - q^n = 0$ can be gotten from

$$p = q(\cos 2\lambda\pi/n \pm \sqrt{-1}\sin 2\lambda\pi/n), \text{ where } \lambda = \pm 1, \pm 2, \cdots.$$

[1] *Histoire d. l'acad. d. sc. et belles let.*, année 1749, Berlin, 1751, pp. 139–179.

Hence

$1 + y/n = \cos 2\lambda\pi/n \pm \sqrt{-1} \sin 2\lambda\pi/n$, and $y = \pm 2\lambda\pi\sqrt{-1}$, when $n = \infty$.

Euler proceeds to find the formulæ for $\log (\pm a)$, $\log (a + b\sqrt{-1})^{\mu/\nu}$, where μ and ν are real integers. Nowhere in this article does he proceed to complex exponents.

Euler treats with remarkable care and clearness the relations between la, $l(-a)$, $2la$, $2l(-a)$, and la^2. He lets p be all even numbers, q all odd numbers. Then

$$l(+a) = A \pm p\pi\sqrt{-1}, \quad l(-a) = A \pm q\pi\sqrt{-1};$$

hence

$$l(+a)^2 = 2l(+a) = 2A \pm 2p\pi\sqrt{-1}, \quad l(-a)^2 = 2l(+a) = 2A \pm 2q\pi\sqrt{-1}.$$

But we have the general formula $la^2 = 2A \pm p\pi\sqrt{-1}$; hence $2l(+a) = la^2$ and $2l(-a) = la^2$, *prenant le signe de = pour marquer, que les valeurs de* $2l(-a)$ *ou de* $2l(+a)$ *se rencontrent parmi les valeurs de* la^2. He adds, *on ne sauroit dire à la verité qu'il soit* $2l(-a) = 2l(+a)$. A superficial comment upon this would be that here things equal to the same thing are *not* equal to each other. The reason why $2l(-a)$ and $2l(+a)$ are not equal is, of course, that $2q$ is never the same number as $2p$, even though both belong to the more general class p. Euler points out another difficulty. If we take $l(-a)^2 = la^2$, in the sense $2A \pm 2q\pi\sqrt{-1} = 2A \pm p\pi\sqrt{-1}$, then we do have $2l(-a) = 2la$, even though, as seen above, la is not $l(-a)$. A superficial comment on this would be that here the axiom—equals divided by equals give equals—breaks down. Euler is dealing here with equations which in the nineteenth century came to be called *incomplete* equations, in which each value on one side of an equation is equal to some value on the other side, but not *vice versa*. Euler proceeds to show under what conditions $2la = 2l(-a)$ becomes a *complete* equation. Let p and p' be even numbers, whether equal or not; let q and q' be odd numbers, whether equal or not. Then take $2l(+a) = 2A \pm (p + p')\pi\sqrt{-1}$, $2l(-a) = 2A \pm (q + q')\pi\sqrt{-1}$; moreover, select the numbers so that $p + p' = q + q'$. Then a complete equation $2l(-a) = 2l(+a)$ is obtained. *Par conséquent dans ce sens on pourra soutenir qu'il est* $2l(-a) = 2l(+a)$, *sans qu'il soit* $l(-a) = l(+a)$. The concept of complete and incomplete equations here created by Euler has not before been attributed to him. Euler did not introduce for the complete equation a special name, nor a special notation. As regards his complete equation $2l(-a) = 2la$, it is readily seen that it involves the condition $p + p' = q + q'$ which is not practical in an algebra of general logarithms. Euler enters into similar discussions for $l(-a)^2$, la^3, la^4.

Euler's fourth problem is, given a logarithm, to find the anti-logarithm x, whether x be real or complex. If the given logarithm is $g\sqrt{-1}$, and g is a multiple of π, then x is 1 or -1. If g is not such a multiple, *pour le trouver on n'a qu'à prendre un arc de cercle* $= g$, *le rayon étant* $= 1$ *& ayant cherché son sinus & cosinus, le nombre cherché sera* $x = \cos g + \sqrt{-1} \sin g$. If the logarithm is

$f + g\sqrt{-1}$, then x is the product of two numbers, one having the logarithm f, the other the logarithm $g\sqrt{-1}$.

We come now to Euler's third paper, *Recherches sur les racines imaginaires des équations*.[1] We have never seen this paper mentioned in historical articles on logarithms. It aims to give two proofs of the theorem that every equation has a root. It was discussed by C. F. Gauss in his memorable inaugural dissertation of 1799.[2] In the second proof Euler shows that expressions involving even complicated arithmetic operations and root extractions can always be brought to the form $m + n\sqrt{-1}$. This part contains new developments on logarithms, but Gauss, in his criticisms of Euler, has no occasion to take up logarithmic theory. Cantor, in his great history,[3] avowedly follows Gauss, hence makes no reference to logarithms. Lampe appears completely to have overlooked this paper of Euler. Nor have we seen any reference to it in the succeeding half century of discussion of logarithmic theory.

Euler generalizes his previous results by considering the general power, in which the base and exponent are both complex. He lets

$$(a + b\sqrt{-1})^{m+n\sqrt{-1}} = x + y\sqrt{-1},$$

where x and y are unknowns to be found. He takes the logarithm of both sides, then differentiates the members of the resulting equation, considering a, b, x and y as variables. Letting $\sqrt{aa + bb} = c$, $A \tan b/a = \phi$, $\sin \phi = b/c$, $\cos \phi = a/c$, he gets, upon equating the reals and the imaginaries,

$$x = c^m e^{-n\phi} \cos (m\phi + nlc),$$

$$y = c^m e^{-n\phi} \sin (m\phi + nlc).$$

Writing in place of ϕ the more general value $\phi + 2\lambda\pi$, he has *toutes les valeurs possibles* which his general power may take, *en donnant à* λ *successivement toutes les valeurs* 0, $\pm$ 1, $\pm$ 2, *etc., où il suffit de prendre pour* c^m *la seule valeur réelle et positive, qui y est renfermée.* We see here that Euler has no hesitation in taking the logarithm of the general power involving complex numbers and that he finds this general power to be infinitely many-valued — a remarkable result at that time.

Proceeding to the consideration of special cases under the general formula just derived, he obtains the result $(\sqrt{-1})^{\sqrt{-1}} = e^{-2\lambda\pi - \pi/2} = 0{,}2078795763507$ for $\lambda = 0$, *qui est d'autant plus remarquable, qu'elle est réelle, et qu'elle renferme même une infinité de valeurs réelles différentes.*

Thereupon, independently of any of his previous results, he devotes half a page to the derivation of the logarithms of a complex number. He assumes $l(a + b\sqrt{-1}) = x + y\sqrt{-1}$, x and y being unknown, then differentiates, and equates the reals and the imaginaries. Thus x and y are found, expressed in terms of arc sine and arc cosine. Here the great question which was

[1] *Histoire de l'acad. r. d. scien. et bell. lett.*, année 1749, à Berlin, 1751, pp. 222–288.
[2] See *Ostwald's Klassiker*, No. 14, Leipzig, 1890; Gauss, *Werke*, Vol. III, 1876, pp. 1–30.
[3] M. Cantor, *op. cit.*, Vol. III, 2. aufl., 1901, pp. 601–604.

agitated for a century was satisfactorily treated within the compass of less than a couple of dozen lines.

Did Euler communicate his results on logarithms to his old and revered master, John Bernoulli I? Historians have given no information on this question. But there is evidence that Euler did so. Pessuti, who took part in the controversy in Italy, states that he met Euler in St. Petersburg and that Euler made to him the statement that "having communicated his [Euler's] results to John Bernoulli shortly before his death, the good old man replied that he died content, because he saw reconciled what seemed to be irreconcilable paradoxes, namely the contradictions which had given rise to the great dispute on logarithms of negative numbers, long continued, between Leibniz and himself."[1] Calandrelli replies to Pessuti that this statement cannot be true, since Bernoulli died in 1748, and Euler's article was sent to the Berlin Academy in 1749. But we know now that Euler had his first paper completed in 1747.

(To be continued.)

AMICABLE NUMBER TRIPLES.[2]

By L. E. DICKSON, University of Chicago.

1. Two numbers are called amicable if each is the sum of the proper divisors of the other, a proper divisor of n being a divisor less than n. We shall say that three numbers form an amicable triple if the sum of the proper divisors of each equals the sum of the remaining two numbers. Let $\sigma(n)$ denote the sum of all the divisors of n. Then n_1, n_2, n_3 form an amicable triple if

$$\sigma(n_1) = \sigma(n_2) = \sigma(n_3) = n_1 + n_2 + n_3. \tag{1}$$

Similarly, $n_1, \cdots, n_k$ form an amicable k-triple if

$$\sigma(n_1) = \sigma(n_2) = \cdots = \sigma(n_k) = n_1 + n_2 + \cdots + n_k. \tag{2}$$

If $k = 2$, n_1 and n_2 are amicable numbers in the usual sense.

For $k > 2$, I have verified that the only solutions of (2) in which not every n_i exceeds 1,000 are those with $k = 3$, $n_1 = n_2 = n_3 = 120$ or 672. When the k numbers n_i are all equal, n_1 is called a multiply perfect number of multiplicity k; as many as 251 such numbers are known,[3] all with $k \leqq 7$.

I obtain below 8 sets of amicable triples in which two of the numbers are equal, and the triple of distinct numbers (end of § 8):

$$293 \cdot 337a, \quad 5 \cdot 16561a, \quad 99371a, \quad a = 2^5 \cdot 3 \cdot 13. \tag{I}$$

Another amicable triple of distinct numbers is obtained in § 14 by a device.

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[1] G. Calandrelli, *Saggio analitico, etc.*, Roma, 1778, p. 14.

[2] Read before the American Mathematical Society, December 31, 1912.

[3] Cf. Carmichael and Mason, *Proceedings Indiana Academy of Science*, 1911, pp. 257–270.

2. Consider an amicable triple apq, ars, at, where p, q, r, s, t are primes not dividing a, and $p \neq q$, $r \neq s$. Then

$$(p+1)(q+1) = (r+1)(s+1) = t+1.$$

Call x the greatest common divisor of $p+1$, $r+1$, and set $p+1 = \alpha x$, $r+1 = \beta x$. Then α, β are relatively prime, and

$$q+1 = \beta y, \quad s+1 = \alpha y, \quad t+1 = \alpha\beta xy,$$

$$\alpha\beta xy\sigma(a) = \sigma(at) = a(pq+rs+t) = a(3\alpha\beta xy - \alpha x - \alpha y - \beta x - \beta y + 1).$$

We therefore set

$$\frac{a}{3a-\sigma(a)} = \frac{b}{c} \quad (b, c \text{ relatively prime}). \tag{3}$$

Hence

$$c\alpha\beta xy - b(\alpha+\beta)(x+y) + b = 0, \tag{4}$$

$$[c\alpha\beta x - b(\alpha+\beta)][c\alpha\beta y - b(\alpha+\beta)] = b^2(\alpha+\beta)^2 - bc\alpha\beta. \tag{5}$$

We assign values to a, α, β, determine b and c by (3), express in all possible ways the second member of (5) as a product of two integers, and take the latter as the values of the two factors on the left of (5). A favorable case is that in which $c\alpha\beta$ is small.

For a even, $p, \cdots, t$ are odd, x and y both even, so that, by (4), b is a multiple of 4. First, let α be even and β odd. Then b is a multiple of 8. For $a = 96$, $b = 8$, $c = 3$, and $\alpha/2$ is odd by (4); for $\alpha = 2$ or 6, $\beta \leqq 19$; $\alpha = 10$ or 14, $\beta \leqq 10$, (5) has no integral solutions except those making $p = 3$, a divisor of a. Next, let α and β be odd. For $a = 96$; $a = 40$ or 224, $b = 4$, $c = 3$; $a = 48$, $b = 12$, $c = 5$, the small values of α, β were examined, but the only solution found is that with $a = 96$, $\alpha = \beta = 1$, viz.,

$$2^5 \cdot 3 \cdot 5 \cdot 43, \quad 2^5 \cdot 3 \cdot 5 \cdot 43, \quad 2^5 \cdot 3 \cdot 263. \tag{II}$$

While this method is less effective than the following one, it introduces naturally the substitution (3), used also below.

3. Let apq, arf, at form an amicable triple, where p, q, r, t are primes not dividing a, and f is an integer relatively prime to a. Call h the greatest common divisor of $\sigma(f)$ and $p+1$, and set $\sigma(f) = gh$, $p+1 = xh$, so that g and x are relatively prime. Then, by (1),

$$r+1 = xy, \quad q+1 = gy, \quad t = ghxy - 1,$$

$$0 = xy[2agh + af - gh\sigma(a)] - a(hx + gy + f).$$

Multiply by b/a and eliminate $\sigma(a)$ by use of (3). We get

$$exy - bhx - bgy - bf = 0, \quad e \equiv bf - bgh + cgh. \tag{6}$$

Multiply (6_1) by egh. We get $MN = T$, where

$$M = h(ex - bg), \quad N = g(ey - bh), \quad T = (ef + bgh)bgh.$$

Solving for x, y, we find that

$$(7) \quad p = \frac{M + b\sigma(f)}{e} - 1, \quad q = \frac{N + b\sigma(f)}{e} - 1, \quad r = \frac{(p+1)(q+1)}{\sigma(f)} - 1,$$

$$(8) \qquad e = bf - b\sigma(f) + c\sigma(f), \quad T = [ef + b\sigma(f)]b\sigma(f).$$

Note that g, h do not occur explicitly in (7), (8). The problem is therefore to assign such relatively prime values to a and f that, when T is expressed as a product MN, the numbers (7) will be integers and primes and

$$(9) \qquad t = (p+1)(q+1) - 1$$

a prime, no one of the four primes to divide a.

With the exception of 945, every odd number $a < 1000$ is deficient, viz., $\sigma(a) < 2a$, and gives $c > b$, whence e is a sum of positive multiples of f and $\sigma(f)$. Thus the denominator e is large compared with f.

Henceforth, we shall assume that a is even. Then p, q, r, f, t are odd, and xy, hx, gy are even. Thus, by (6), b is even. If x, y are not both even, h or g is even and e is even. Hence in every case, exy and therefore also b is divisible by 4. Then by (3), a is either a multiple of 8 or is the product of 4 and an odd square.

4. We have shown that, when a is even, $b = 2^\beta B$, where $\beta \geqq 2$ and B is odd. We proceed to prove the noteworthy and very useful

THEOREM: *There is no amicable triple with $\sigma(f)$ divisible by 2^β.*

Suppose that $\sigma(f) = 2^\beta \phi$. Write γ for the odd number $b - c$. Then by (8)

$$(10) \qquad e = 2^\beta k, \quad k \equiv Bf - \gamma\phi,$$

$$(11) \qquad MN = 2^\beta A \cdot 2^\beta B\sigma(f), \quad A \equiv kf + B\sigma(f).$$

First, let ϕ be even. Then k and A are odd. By (7), M and N are multiples of $2 \cdot 2^\beta$, since p and q are odd. Thus

$$M = 2^{\beta+1}m, \quad N = 2^{\beta+1}n, \quad 4mn = AB\sigma(f),$$

$$k(p+1) = 2m + B\sigma(f), \quad k(q+1) = 2n + B\sigma(f),$$

$$k^2(r+1) = \frac{4mn}{\sigma(f)} + E = AB + E, \quad E \equiv 2B(m+n) + B^2\sigma(f).$$

Since E is even, $r + 1$ is odd, whereas r is to be an odd prime.

Next, let ϕ be odd. Then $k = 2^\kappa K$, $\kappa \geqq 1$, K odd. By (7),

$$(12) \qquad 2^{\beta+\kappa}K(p+1) = M + 2^{2\beta}B\phi.$$

Consider the sub-case $\kappa \geqq \beta - 1$. Since p is odd, the left member of (12) is divisible by $2^{2\beta}$. The same is true of M and of N. Set

$$M = 2^{2\beta}m, \quad N = 2^{2\beta}n.$$

By (11), A and therefore also k is divisible by 2^β. Thus

$$\kappa \geqq \beta, \quad A = 2^\beta\alpha, \quad \alpha \equiv 2^{\kappa-\beta}fK + B\phi.$$

By (12),

$$0 \equiv 2^{2\beta}(m + B\phi)(\text{mod } 2^{\beta+\kappa+1}), \quad m \equiv - B\phi(\text{mod } 2^{\kappa+1-\beta}).$$

From the latter and the like congruence for n, we get

$$mn \equiv B^2\phi^2(\text{mod } 2^{\kappa+1-\beta}).$$

But, by (11), $mn = \alpha B\phi$. Replacing α by its value, we get

$$2^{\kappa-\beta}fKB\phi + B^2\phi^2 = mn \equiv B^2\phi^2(\text{mod } 2^{\kappa+1-\beta}),$$

which is impossible since $fKB\phi$ is odd.

Finally, let $\kappa < \beta - 1$. By (12), M is divisible by $2 \cdot 2^{\beta+\kappa}$. Let

$$M = 2^{\beta+\kappa+1}m, \quad N = 2^{\beta+\kappa+1}n, \quad \alpha = fK + 2^{\beta-\kappa}B\phi.$$

Then α is odd. By (11),

$$A = 2^\kappa\alpha, \quad mn = 2^{\beta-\kappa-2}\alpha B\phi.$$

By (12) or (7),

$$K(p + 1) = 2m + 2^{\beta-\kappa}B\phi, \quad K(q + 1) = 2n + 2^{\beta-\kappa}B\phi,$$

$$K^2(r + 1) = \frac{4}{2^\beta\phi}(m + 2^{\beta-\kappa-1}B\phi)(n + \cdots) = \frac{1}{2^\kappa}[\alpha B + 2B(m + n) + 2^{\beta-\kappa}B^2\phi].$$

The number in brackets is odd. Hence $r + 1$ is not an integer.

For the case f a prime, the theorem yields the

COROLLARY: There is no amicable triple apq, arf, at, where p, q, r, f, t are primes not dividing the even number a, and at least one of the primes p, q, r, f is of the form $2^\beta\phi - 1$, where 2^β is the highest power of 2 dividing b, defined by (3).

5. Applying the theorem for $\beta = 2$ or 3, and finding the solutions f of $\sigma(f) = w, 2w, 4w$, where w is odd, we obtain the

THEOREM. *If $b = 4B$, B odd, an amicable triple has*

$$f = (4k + 1)^{4l+1}g^2,$$

where $4k + 1$ is unity or a prime not dividing g. If $b = 8B$, f may have also one of the following values

$$(8k + 3)^{4l+1}g^2, \quad (4k + 1)^{8l+3}g^2, \quad (4k + 1)^{4l+1}(4j + 1)^{4v+1}g^2,$$

where $8k + 3$, $4k + 1$, $4j + 1$ are primes not dividing g.

6. The case $c = 1$ is favorable since e is then small in comparison with f. The minimum b is then 4. The condition for $b = 4$, $c = 1$ is, by (3), $11a = 4\sigma(a)$.

The least k for which $\sigma(2^k)$ has the factor 11 is $k = 9$. Let $a = 2^9\alpha$, α odd. Then $2^7\alpha = 3 \cdot 31\sigma(\alpha)$. Set $\alpha = 31\gamma$. Then $3\sigma(\gamma) = 4\gamma$, which holds for $\gamma = 3$. Thus a solution is $a = 2^9 \cdot 3 \cdot 31$. For $f = 1$, we obtain the solution

(III) $\quad 5 \cdot 13a, \; 83a, \; 83a, \quad a = 2^9 \cdot 3 \cdot 31.$

Next, let f be a prime. By § 5, $f = 4l + 1$. Then $e = 2(2l - 1)$. Thus

$$M = 4m, \quad mn = (2l + 1)(8l^2 + 6l + 3), \quad (2l - 1)(p + 1) = 2(m + 4l + 2).$$

Hence m and n are $\equiv -4 \pmod{2l - 1}$. For $f = 5$, $m = 3$, $n = 17$, $q = 45$. For $f = 13$, $m = 1$, $n = 3 \cdot 7 \cdot 31$, $q = 265$, or $m = 21$, $n = 31$, which yields the solution

(IV) $\quad 13 \cdot 17a, \; 13 \cdot 17a, \; 251a, \quad a = 2^9 \cdot 3 \cdot 31.$

For $f = 37$, $m = 3 \cdot 5 \cdot 19$, $n = 47$, $q = 9$. For $f = 137$, $m = 3 \cdot 5 \cdot 31$, $n = 23 \cdot 61$, $q = 45$. For the remaining primes $4l + 1 < 200$, there is no set of values m, n; likewise, not for $f = 7^2, 11^2, 17^2$. For $f = 5^2$, $e = 7$, $M = 2m$, $m \equiv 1 \pmod 7$, $m = 1$, $n = 13 \cdot 23 \cdot 31$, $q = 2665$. For $f = 13^2$, $e = 127$, $m \equiv 15 \pmod{127}$, $m = 15$, $n = 23 \cdot 61 \cdot 193$, $q \equiv 0 \pmod 3$. The smallest $f = v^2w^2$ (v, w distinct primes) making $e > 0$ are $5^2 \cdot 17^2$ and $7^2 \cdot 11^2$, giving $e = 349$ and 973, respectively. Each prime factor of f exceeds 3 by the

THEOREM. *Each prime factor of f exceeds $b/c - 1$.*

Proof is needed only when $b > c$. If v is a prime factor of f, we have, since $e > 0$,

$$\sigma(f) \geqq f + \frac{f}{v}, \quad bf > (b - c)\left(f + \frac{f}{v}\right), \quad c > \frac{(b - c)}{v}.$$

7. Let $b = 12$, $c = 1$. The condition (3) is $12\sigma(a) = 35a$. For $a = 2^3\alpha$, α odd, $3^2\sigma(\alpha) = 2 \cdot 7\alpha$; let $\alpha = 3^2\gamma$; then $13\sigma(\gamma) = 14\gamma$, which holds for $\gamma = 13$. For $a = 2^5\alpha$, $3^3\sigma(\alpha) = 2^3 \cdot 5\alpha$, which holds for $\alpha = 3^3$.

For $f = 1$, we have $e = 1$, $M = 2m$, $mn = 3 \cdot 13$, $m = 1$ or 3,

$$(p, q, r) = (13, 89, 1259) \text{ or } (17, 37, 683),$$

all primes, and $t = r$. We thus have the three amicable triples

(V) $\quad 13 \cdot 89a, \; 1259a, \; 1259a, \quad a = 2^5 \cdot 3^3,$

(VI), (VII) $\quad 17 \cdot 37a, \; 683a, \; 683a, \quad a = 2^5 \cdot 3^3$ or $2^3 \cdot 3^2 \cdot 13$.

Let f be a prime (> 11, by § 6). Then $f = 4k + 1$, $M = 4m$, $N = 4n$, $e = 4k - 10$, $mn = 3(2k + 1)(8k^2 + 6k + 7)$, $(2k - 5)(p + 1) = 2m + 12(2k+1)$. Thus $m \equiv -36 \pmod{2k - 5}$. If $k = 9j + 7$, $2k - 5 = 9(2j + 1)$, $m = 9m'$, $n = 9n'$, $m'n' = (6j + 5)(72j^2 + 118j + 49)$, $(2j + 1)(p + 1) = 2m' + 4(6j + 5)$, $m' \equiv -4 \pmod{2j + 1}$. For $j = 0$, $f = 29$, $m'n' = 5 \cdot 7^2$, while $m' = 1$, 7 or 7^2 gives $p \equiv 0 \pmod 3$. For $j = 1$, f is composite. For $j = 2$, $f = 101$, $m'n' = 3 \cdot 17 \cdot 191$; either $m' = 1$, $q = 3 \cdot 1303$, or $m' = 51$, $p = 3 \cdot 11$.

For $j = 3$, $f = 137$, there is no set m, n. For $j = 4$, $f = 173$, $m' = 7 \cdot 29$, $n' = 239$, $p = 3 \cdot 19$. Next, if $k = 3v + 1$, $v \not\equiv 3j + 2$, $m = 3m'$ and

$$m'n' = 3(2v+1)(24v^2 + 22v + 7), \quad (2v-1)(p+1) = 2m' + 12(2v+1),$$
$$m' \equiv -12 (\text{mod } 2v - 1).$$

For $v = 1, f = 17$, $m'n' = 9 \cdot 53$; if $n' = 53$ or $9 \cdot 53$, $q = 3 \cdot 47$ or $23 \cdot 43$; if $n' = 3 \cdot 53, p = 41, q = 353$, but $r = 825$. For $v = 3, f = 41, m'n' = 3 \cdot 7 \cdot 17^2$, $m' = 3$, $p = 17$, $q = 825$. For $v = 4, f = 53$, $m'n' = 3^3 \cdot 479$, $m' = 9$, $p = 17$, $q = 425$. For $v = 6$, 10, 13, 15, f is composite. For $v = 7$, $f = 89$, $m'n' = 3^2 \cdot 5 \cdot 7 \cdot 191$; either $m' = 1$, $p = 13$, $q = 13 \cdot 23 \cdot 31$ or $m' = 3 \cdot 5 \cdot 7$, $p = 29$, $q = 101$, $r = 33$. For $v = 9, f = 113$ and for $v = 16, f = 197$, there is no set m, n.

For $k \not\equiv 1 (\text{mod } 3)$ the primes $f < 200$ for which there occur sets m, n are $f = 13, 37, 97$. For $f = 13$, $mn = 3 \cdot 7 \cdot 97$; if $m = 1$ or 21, p has the factor 5; if $m = 3$ or 7, q has the factor 11 or 5. For $f = 37$, $m = 3$, $n = 19 \cdot 709$, $p = 17$ and $q = 2089$ are primes, but $r = 23 \cdot 43$. For $f = 97$, $m = 7$, $n = 3 \cdot 7 \cdot 4759$, $p = 13$, $q = 59 \cdot 79$.

Next, let $f = v^2$, v a prime > 11. For $v = 13$, $M = 6m$, $mn = 3 \cdot 19 \cdot 61 \cdot 83$, either $m = 19$, $p = 3 \cdot 51$ or $m = 19 \cdot 61$, $p = 609$. For $v = 19$, $M = 6m$, $mn = 3 \cdot 11 \cdot 41^2 \cdot 127$, $n = 41 \cdot 127$, $q = 11 \cdot 23$. For $v = 17$ or 23, there is no set m, n.

For $f = v^2w^2$, where v and w are distinct primes $\leqq 29$, e is positive only for $v = 23$, $w = 29$, and $e = 40375$.

8. A very favorable case is that in which $b\sigma(f)$ and hence M and N are multiples of e, since the computed values of p, q are always integers. This case does not arise for $f = 1$, whence $\sigma = 1$, $e = c$, since c is prime to b. Let f be a prime. Then $e = c\sigma - b$. Let g be the greatest common divisor of $b = g\beta$ and $\sigma = g\phi$, so that ϕ is prime to β and hence to $E = c\phi - \beta$. Since $e = gE$ divides $gb\phi$, E divides b and therefore also g. Let $g = EQ$. Then

$$b = \beta EQ, \quad \sigma(f) = \phi EQ, \quad e = E^2Q.$$

Conversely, this e divides $b\sigma(f)$.

First, let $f = 5$. Since $\sigma = 6$ and b is a multiple of 4, $g = 2$ or 6, and β is even, E odd. If $g = 6$, $\phi = 1$, $c - \beta = E = 1$ or 3, $b = 6\beta$. In the latter case, $\beta = c - 3$, and c is prime to 6. For $c = 5$, $b = 12$ and we may take $a = 48$, an unfruitful case (§ 2). For $c = 7$, $b = 24$ and we may take $a = 72$. Then for $f = 5$, $e = 18$, $M = 36m$, $mn = 26$, $p + 1 = 2m + 8$; if $m = 1$, $p = 3^2$; if $m = 2$, $p = 11$, $q = 3 \cdot 11$. For $f = 11$, $e = 60$, $M = 24m$, $mn = 6 \cdot 79$, $p + 1 = (2m + 24)/5$, $m \equiv 3 (\text{mod } 5)$, $m = 3$, $p = 5$, $q = 67$, $r = 3 \cdot 11$. For $f = 13, 17, 19, 25$, there is no set m, n.

Next, let $g = 2$. Then $\phi = 3$, $E = 1$, $b = 2\beta$, $\beta = 3c - 1$. If $c = 1$, $b = 4$, a case treated in § 6. If $c = 3$, $b = 16$ and (3) gives $45a = 16\sigma(a)$. The simplest solution arises from $a = 2^7\alpha$, α odd. Then $17\sigma(\alpha) = 8 \cdot 3\alpha$. Let $\alpha = 17\gamma$, γ prime to 17. Then $3\sigma(\gamma) = 4\gamma$, $\gamma = 3$, whence $a = 2^7 \cdot 3 \cdot 17$.

For $f = 1$, $e = 3$, $M = 2m$, $mn = 4 \cdot 19$, either $m = 1$, $p = 5$, $q = 55$ or $m = 4$, $p = 7$, $q = 17$, $r = 11 \cdot 13$. For $f = 5$, $e = 2$, $M = 4m$, $mn = 2^2 \cdot 3 \cdot 53$, $p + 1 = 2m + 48$, $m \not\equiv 2 \pmod 3$, whence either $m = 1$, $p = 7^2$ or $m = 6$, $p = 59$, $q = 3 \cdot 51$. For $f = 7$, $e = 8$, $M = 16m$, $mn = 4 \cdot 23$, $p + 1 = 2m + 16$, whence

m	p	q	r	t
1	17	119	449	$59 \cdot 61$
2	19	107	269	$17 \cdot 127$
4	23	61	$5 \cdot 37$	

For $f = 11$, $e = 20$, $M = 8m$, $mn = 4 \cdot 3 \cdot 103$, either $m = 1$, $p = 9$ or $m = 6$, $p = 11$, $q = 7 \cdot 13$. For $f = 17$, $e = 38$, $M = 4m$, $m = 4$, $n = 3^2 \cdot 467$, $p = 7$, $q = 449$, $r = 199$, $t = 59 \cdot 61$. For $f = 19$, $e = 4 \cdot 11$, $m = 4$, $n = 5 \cdot 17^2$, $p = 7$, $q = 269$, $r = 107$, $t = 17 \cdot 127$. For $f = 23$, $e = 8 \cdot 7$, $M = 16m$, $mn = 4 \cdot 3 \cdot 11 \cdot 19$, either $m = 4$, $p = 7$, $q = 185$, or $m = 11$, $p = 9$. For $f = 29$, $M = 4m$, $m = 5 \cdot 13$, $n = 2^2 \cdot 3 \cdot 101$, $p = 9$. For $f = 13$ or 7^2, there is no set m, n.

If $c = 5$ or 7, the solution of (3) is not evident. Let $c = 9$. Then $b = 4 \cdot 13$, $4 \cdot 13\sigma(a) = 3 \cdot 7^2 \cdot a$. Let $a = 13\alpha$, α not divisible by 13. Then $2^3\sigma(\alpha) = 3 \cdot 7\alpha$. Let $\alpha = 2^5\gamma$. Then $3\sigma(\gamma) = 4\gamma$, $\gamma = 3$, $a = 2^5 \cdot 3 \cdot 13$. For $f = 1$, $e = 9$, $M = 2m$, $m = 1$, $n = 13 \cdot 61$, giving the amicable triple

$$\text{(VIII)} \qquad 5 \cdot 181a, \quad 1091a, \quad 1091a, \quad a = 2^5 \cdot 3 \cdot 13.$$

For $f = 5$, $e = 2$, $M = 4m$, $mn = 3 \cdot 7 \cdot 13 \cdot 23$, $p + 1 = 2(m + 78)$, $m \not\equiv 2$, $n \not\equiv 2 \pmod 3$. We may take n a multiple of $3 \cdot 23$. For $m = 1$, $p = 157$ and $q = 12713$ are primes, but $r = 53 \cdot 6317$. For $m = 7$, $p = 13^2$. For $m = 13$, $p = 181$, $q = 19 \cdot 59$. For $m = 7 \cdot 13$, we obtain solution I of § 1. For $f = 13$ or 17, there is no set m, n.

9. In the intial work of § 8, let $f = 17$. Then $g = 2$, 6 or 18. To make $e = gE$ a minimum, we take $E = 1$. Then for $g = 18$, $\phi = 1$, $b = 18\beta$, $\beta = c-1$, c prime to 18. For $c = 5$, 7 or 11, the solution of (3) is not evident. For $c = 13$, $b = 2^3 \cdot 3^3$, $5 \cdot 127a = 2^3 \cdot 3^3\sigma(a)$. Take $a = 2^6\alpha$, α odd. Then $2^3 \cdot 5\alpha = 3^3\sigma(\alpha)$, $\alpha = 3^3$. For $f = 1$, $e = 13$, $M = 2m$, $m = 9$, $n = 2 \cdot 3 \cdot 229$, $p = 17$ and $q = 227$ are primes, but $r = 11 \cdot 373$. For f a prime, $e = 13f - 203$, $f > 14$. For $f = 17$, $e = 18$, $M = 36m$, $mn = 2 \cdot 3^3 \cdot 233$, $p + 1 = 2m + 216$, $m \not\equiv 2 \pmod 3$, $m \neq 2$, $n \neq 233$. For $m = 1$ or 3, p is composite. For $m = 27$, $q = 31 \cdot 37$. For $m = 6$, 9 or 18, p and q are primes, but r is composite. For $f = 19$, $M = 8m$, either $m = 10$, $p = 99$ or $m = 54$, $n = 5 \cdot 1289$, $q = 3 \cdot 423$. Finally, for $c = 17$, $b = 2^5 \cdot 3^2$, $7 \cdot 11^2a = 2^5 \cdot 3^2\sigma(a)$. Let $a = 2^5\alpha$, α odd. Then $11^2\alpha = 3^4\sigma(\alpha)$, $\alpha = 3^4$. For $f = 1$, $e = 17$, $M = 2m$, $mn = 2^3 \cdot 3^2 \cdot 5 \cdot 61$, $p + 1 = 2(m + 144)/17$, either $m = 9$, $p = 17$, $q = 303$ or $m = 2^2 \cdot 3 \cdot 5$, leading to the amicable triple

$$\text{(IX)} \qquad 23 \cdot 59a, \quad 1439a, \quad 1439a, \quad a = 2^5 \cdot 3^4.$$

For f a prime, $e = 17f - 271$, $f > 14$. If $f = 17$, $e = 18$, $M = 36m$, $mn = 2^3 \cdot 3^2 \cdot 5 \cdot 61$, $p + 1 = 2m + 288$, m, $n \not\equiv 2 \pmod 3$. There are 16 cases; but p is a prime only for $m = 10, 40, 3, 12, 36, 15, 30, 90$. If $m = 10$, $q = 4679$, $r = 3 \cdot 26693$. If $m = 40, 3, 36, 15, 30, 90$, q is composite. If $m = 12$, $q = 3947$, $r = 11 \cdot 6221$. For $f = 19$, $e = 4 \cdot 13$, $M = 8m$, $mn = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 241$, $p + 1 = 2(m + 5)/13 + 110$, $m \equiv 8 \pmod{13}$. There are 4 m's. If $m = 8$ or 60, $p = 3 \cdot 37$ or $7 \cdot 17$. If $m = 21$, $p = 113$, $q = 47 \cdot 97$. If $m = 2^3 \cdot 3 \cdot 5 \cdot 7$, $p = 239$, $q = 13 \cdot 17$. For $f = 23$, $e = 120$, $M = 48m$, $mn = 2^3 \cdot 3^2 \cdot 13 \cdot 31$, $p + 1 = 2(m + 144)/5$, $m \equiv 1 \pmod 5$. There are 6 m's. If $m = 1$, $p = 3 \cdot 19$. If $m = 2^2 \cdot 3 \cdot 13$, $p = 7 \cdot 17$. If $m = 6$, $q = 11 \cdot 181$. If $m = 2^3 \cdot 3^2 \cdot 13$, $q = 3 \cdot 23$. If $m = 2^2 \cdot 3^2$, $p = 71$, $q = 379$, $r = 17 \cdot 67$. If $m = 26$, $p = 67$, $q = 503$, $r = 1427$ are primes, but $t = 43 \cdot 797$.

10. For $a = 144 = b$, $c = 29$, $f = 5$, $e = 30$, $M = 12m$, $mn = 2^2 \cdot 3^2 \cdot 13^2$, $p + 1 = (2m + 144)/5$, $m \equiv 3 \pmod 5$; $m = 2 \cdot 3^2$ or 13 gives $p = 35$ or 33; $m = n = 2 \cdot 3 \cdot 13$ gives $p = q = 59$, $r = 599$, $t = 59 \cdot 61$; $m = 3$, gives $p = 29$, $q = 839$, $r = 13 \cdot 17 \cdot 19$. For $f = 1, 7, 11$, or 25, there is no set m, n. For $f = 125$, $e = 60$, $M = 24m$, $mn = 2^2 \cdot 3^2 \cdot 11 \cdot 13 \cdot 127$, $p + 1 = 2(m + 936)/5$, $m \equiv 4 \pmod 5$, so that $m = 4, 9, 3 \cdot 13$ or $2 \cdot 3^2 \cdot 13$ or their products by 11. Then p is composite except for $m = 3 \cdot 13$, $q = 17 \cdot 727$, and for $m = 2 \cdot 3^2 \cdot 13$, $p = 467$, $q = 2371$, $r \equiv 0 \pmod 5$.

11. For $a = 192 = 2^6 \cdot 3$, $b = 48$, $c = 17$. If $f = 5$, $e = 54$, $M = 36m$, $mn = 4 \cdot 31$, $p + 1 = 2(m + 8)/3$; either $m = 1$, $q = 3 \cdot 29$ or $m = 4$, $q = 25$. If $f = 7$, $e = 8 \cdot 11$, $M = 16m$, $mn = 2^2 \cdot 3 \cdot 5^3$, $p + 1 = 2(m + 24)/11$, $m = 20$, $p = 7$, $q = 17$, $r = 17$, $t = 11 \cdot 13$. If $f = 35$, $e = 64 \cdot 3$, $M = 128 \cdot 3m$, $mn = 3 \cdot 47$, $p + 1 = 2m + 12$; if $m = 1$, $p = 13$, $q = 293$, but $r + 1$ is not an integer; if $m = 3$, $q = 105$. If $f = 55$, $e = 8 \cdot 51$, $M = 48m$, $mn = 2^2 \cdot 3^2 \cdot 13 \cdot 83$, $m = 13$ or $2 \cdot 3^2 \cdot 13$, $p = 9$ or 35. If $f = 1, 11, 13$, or 25, there is no set m, n.

12. For $a = 2^7 \cdot 3$, $b = 32$, $c = 11$. If $f = 5$, $e = 34$, $M = 4m$, $mn = 2^3 \cdot 3 \cdot 181$, $m = 3$, $q = 175$. If $f = 7$, $e = 8 \cdot 7$, $M = 16m$, $mn = 2^3 \cdot 3^4$, $m = 2^2 \cdot 3$, $p = 7$, $q = 19$, $r = 19$, $t = 3 \cdot 53$. If $f = 11$, $p = 3$, excluded. If $f = 35$, $e = 16 \cdot 7$, $M = 32m$, $mn = 2^3 \cdot 3 \cdot 11 \cdot 31$, $p + 1 = 2(m + 48)/7$; $m = 1$, $p = 13$, $q = 2351$, $r = 685$; $m = 8$, $p = 15$; $m = 22$, $q = 7 \cdot 17$. For $f = 1, 13, 55$ or 175, there is no set m, n.

13. For $a = 2^4 \cdot 3^2 \cdot 17$, $b = 8 \cdot 17$, $c = 5$. If $f = 1$, $e = 5$, $M = 2m$, $mn = 2 \cdot 3 \cdot 17 \cdot 47$, $m \equiv 2 \pmod 5$; $m = 2$, $p = 27$; $m = 17$, $p = 33$; $m = 6 \cdot 17$, $p = 67$, $q = 45$. If f is a prime, $e = 5f - 131$, $f > 26$. For $f = 29$, $e = 14$, $M = 4m$, $mn = 2 \cdot 3 \cdot 5 \cdot 17 \cdot 2243$, $m \equiv 2 \pmod 7$; $m = 2$, $p = 3 \cdot 97$; $m = 30$, $p = 13 \cdot 23$; $m = 3 \cdot 17$, $p = 305$; $m = 2 \cdot 5 \cdot 17$, $p = 3 \cdot 113$. For $f = 37$, 41, 43, or 29^2, there is no set m, n.

14. We may employ a simple tentative method of finding amicable triples ar, as, at, where a is relatively prime to r, s, t. Let r, s, t be chosen so that $\sigma(r) = \sigma(s) = s(t)$. The problem is to find a such that also

$$\sigma(r) \cdot \sigma(a) = (r + s + t)a.$$

The set r, s, t is to be rejected if $\sigma(r)$ has a prime factor dividing one of the numbers $r, s, t,$ but not their sum, since it must then divide a. All such sets in which the common divisor sum $\sigma(r)$ does not exceed 360 are given in the following list

$$
\begin{array}{rrrr}
2^2 \cdot 11, & 5 \cdot 13, & 83, & 7\sigma(a) = 2^4a; \\
2 \cdot 5 \cdot 7, & 5 \cdot 23, & 7 \cdot 17, & 3^2\sigma(a) = 19a; \\
5 \cdot 31, & 7 \cdot 23, & 191, & 2^6\sigma(a) = 13^2a; \\
7 \cdot 29, & 11 \cdot 19, & 239, & 2^4 \cdot 5\sigma(a) = 7 \cdot 31a; \\
2 \cdot 5 \cdot 13, & 5 \cdot 41, & 13 \cdot 17, & 3^2 \cdot 7\sigma(a) = 139a; \\
5 \cdot 41, & 13 \cdot 17, & 251, & 2^2 \cdot 3^2 \cdot 7\sigma(a) = 677a; \\
2 \cdot 83, & 5 \cdot 41, & 13 \cdot 17, & 3^2 \cdot 7\sigma(a) = 2^2 \cdot 37a; \\
3 \cdot 89, & 11 \cdot 29, & 17 \cdot 19, & 2^3 \cdot 5\sigma(a) = 101a; \\
11 \cdot 29, & 17 \cdot 19, & 359, & 2^3 \cdot 3^2 \cdot 5\sigma(a) = 7 \cdot 11 \cdot 13a; \\
3 \cdot 89, & 11 \cdot 29, & 359, & 2^3\sigma(a) = 3 \cdot 7a.
\end{array}
$$

For the final equation the solution $a = 2^5 \cdot 3$ obtained at the end of § 8 is now excluded since 3 divides one of the triple. A valid solution may be found tentatively. Let $a = 2^nb$. For certain values of $n < 14$, a would necessarily be divisible by one of the prime factors of the numbers of the triple; for the others, a solution seems improbable. Take $n = 14$. Then $31 \cdot 151\sigma(b) = 2^{11} \cdot 3b$. Let $b = 31 \cdot 151c$. Then $19\sigma(c) = 2^3 \cdot 3c$. Let $c = 19d$. Then $5\sigma(d) = 2 \cdot 3d$, $d = 5$. Hence we have the amicable triple

$$
\text{(X)} \qquad 3 \cdot 89a, \quad 11 \cdot 29a, \quad 359a, \quad a = 2^{14} \cdot 5 \cdot 19 \cdot 31 \cdot 151.
$$

MATHEMATICAL LITERATURE FOR HIGH SCHOOL TEACHERS.[1]

By G. A. MILLER, University of Illinois.

There is no sharp line of division between the mathematics for the high school teacher and that which is more advanced. All teachers should strive to increase their knowledge of the subjects which they teach. The teacher who does not hunger and thirst for more knowledge cannot hope to inspire his students with the proper zeal and earnestness. The desire for more knowledge should be based, in part, on the fact that we seek more light on the subjects which we teach.

Mathematics Not a Finished Science.—Even such elementary concepts as those of the natural numbers become more replete with meaning if we study the efforts to base these concepts on those of the theory of aggregates. Questions relating directly to the natural numbers have been considered in recent years by some of the foremost mathematicians, especially in view of the fact that the

[1] Address before the mathematical section of the High School Conference held at the University of Illinois in November, 1912. Reported by Dr. W. W. Denton.

contradictions which have appeared in the theory of infinite aggregates raise the question whether the concept of natural numbers should be based on this theory.[1] This reference may suffice to emphasize the fact that even elementary mathematics is not a finished science, but one which is continually being enriched through the developments in the various fields of our subject.

The Teacher Cannot Stand Still.—Another reason why teachers should strive to increase their knowledge is based on the fact that our intellectual standards are being rapidly raised. This applies to the teachers in the universities as well as to those in the high schools. The fact that a teacher knows enough to meet the demands made upon him to-day does not imply that an equal amount of knowledge will suffice ten years from now. It is well known that some of the men who were appointed as professors in the universities twenty years ago could not meet the requirements of an instructorship to-day. Since most high schools are less conservative than the universities as regards the dismissal of those who do not meet the requirements of the time, it would appear that the high school teacher has special reasons to observe the drift of things.

The Rapid Growth of Libraries.—One evidence of our rapid intellectual advance is furnished by the great growth of our libraries and their facilities to serve the people. The Illinois State University Library, for instance, has doubled the number of its volumes within a decade and is continually becoming a stronger factor in the intellectual advance of the state. About a score of cities in our country contain libraries whose annual income exceeds $100,000, and the city of Chicago has at least three such libraries. While our libraries are not yet as large as some of those in Europe, we have now at least three containing more than a million volumes each, namely, the libraries of Congress, New York and Boston.

The great development of our libraries implies an active intellectual growth of the people as a whole, and with this growth there naturally comes a demand for higher and better standards on the part of our teachers. To attain these standards it is necessary that we grow, and the most useful aid to intellectual growth is literature. The mathematical literature may be divided into two large classes,—periodic and non-periodic. These classes we shall call journals and books respectively.

Early Mathematical Journals.—The journals are continually assuming a more and more important place in our literature; and they vary so much that almost all classes of readers are now provided for. The first American mathematical journal, known as the *Mathematical Correspondent*, was started about a century ago, but it did not survive long. Several other journals had a similar fate. Our oldest extant mathematical journal, known as the *American Journal of Mathematics*, is less than forty years old, having been started in 1878.

Even in Europe mathematical journals are of a comparatively recent origin. Hence we find ourselves near the beginning of mathematical journals, but we

[1] Cf. C. Färber, "Der Zahlbegriff in Lehrbüchern und im Unterricht," *Archiv der Mathematik und Physik*, vol. 20 (1912), p. 139.

have reason to believe that many of those now extant will live forever. It is interesting to consider the effect of these journals on the future development of our science. As a rule each volume marks an advance over its predecessor, and the 142 volumes of the German journal called, *Journal für die reine und angewandte Mathematik,* furnish some evidence with respect to the elevating influence of the journals of high grade.

Journals for Secondary Teachers.—The most elementary American regular periodicals which are largely or entirely devoted to mathematics are *School Science and Mathematics* and *The Mathematics Teacher.* The AMERICAN MATHEMATICAL MONTHLY is of a little more advanced type, but most of its articles and problems should be within the easy reach of the average high school teacher. The English journal called *Mathematical Gazette* is the organ of their Mathematical Association, formerly known as the "Association for the improvement of geometric teaching." This association aims to bring within its consideration all branches of elementary mathematics, and its journal is of about the same grade as the AMERICAN MATHEMATICAL MONTHLY. The *Annals of Mathematics,* now published under the auspices of Princeton University, is of a somewhat higher grade, but it should be read by the more advanced high school teachers.

Each of the other leading mathematical languages, French, German, and Italian, has similar journals; and these journals are playing an important rôle in the mathematical development of the people using these languages. This implies that periodic literature is filling a want of the teachers of the high school grade in all the leading countries of the world. We are naturally led to inquire why journals are so desirable, and what useful objects they accomplish.

The Influence of Journals.—In the first place, journals bring us face to face with many of the modern questions. They facilitate collaboration and represent the dynamic elements in mathematical literature, while books represent more nearly the static. The following instance may serve to illustrate the point in question. About a year and a half ago a mathematical journal called *Revista de la Sociedad Matemática Española* was started at Madrid, and twenty-five Spanish mathematicians attended the Fifth International Congress of Mathematicians during last summer, while the largest attendance of Spanish mathematicians at any of the preceding congresses was five. Journals seem to accompany the mathematical activity of nations, and hence it is reasonable to assume that they exercise a helpful influence.

It is, of course, true that books have many advantages. They are suitable for the more extended and systematic presentation of subjects and for grading the work so as to put it into a form for the beginner. The point which we desire to emphasize is that the modern teacher needs journals as well as books, and that he should not fail to utilize the advantages offered by the best periodic literature. The loyal support rendered to this literature will tend to improve its quality and to increase its influence for good. We should all aim to enhance the dignity of our profession and to show our interest in it by exerting some influence along lines that tend to ennoble it.

Books of Reference.—While it is comparatively easy to suggest the most suitable mathematical journals, it is much more difficult to give equally good advice with respect to books, since the number of such books is so large. All that we shall aim to do in this connection is to call attention to a few books of reference and to a few works of a general mathematical character.

In 1908 the Royal Society of London published the volume on pure mathematics of its *Subject Index* of the scientific papers which appeared during the nineteenth century. This volume includes 38,748 entries, relating to the mathematical papers which appeared in 700 different serials. It seems to me that all students of mathematics should have access to such reference books, since they serve as a reliable guide to the literature on a large number of different mathematical subjects, and they enable one to get a correct general view of the developed parts of our science. Hence they tend to eliminate the waste involved in doing the same thing over and over without utilizing the results attained by others.

The Great Encyclopedias.—The most valuable modern work of reference, especially for the advanced student of mathematics, is the large mathematical encyclopedia which is being published in German and in French. Many parts of this work relate to subjects which should be of interest to high school teachers. Unfortunately, the parts relating to teaching, history and philosophy are to appear at the end and have not yet been completely planned. The extent of this work may be inferred from the fact that 35 volumes of the French edition have already been planned. It is hoped that the first volume of this edition, devoted to arithmetic, can be completed in the near future. The first volume of the German edition was completed several years ago and it is already out of print.

Valentin's Bibliography.—Another great work of reference, which has been in course of preparation for more than 25 years, should interest all students of mathematics. In 1885 G. Valentin of Berlin, Germany, began to prepare a bibliography of mathematical literature, which is to give a list of all the books and articles that appeared up to 1900. He collected about 150,000 titles, and about one-fourth of these are titles of books, counting different editions and translations of the same book as a single book. This is the most extensive list of mathematical publications that has ever been made, and it is expected that a part of it will be ready for publication during the coming year.

Other General Works.—Those who read English only will doubtless find Carr's *Synopsis of Elementary Results in Pure Mathematics* very useful as a work of reference, even if it is a little old. Among the most modern books in the English language dealing with general mathematics of a high school grade, we may mention the following four: "Monographs on topics of modern mathematics relevant to the elementary field," edited by J. W. A. Young; "Lectures on fundamental concepts of algebra and geometry," by J. W. Young; "Introduction to mathematics," by A. N. Whitehead; and "Easy mathematics, chiefly arithmetic," by Sir Oliver Lodge.

The History of Mathematics.—During recent years there has been a rapidly growing interest in the history of mathematics. This has led to a re-examination

of many matters which had been accepted as historical facts. In a fairly large number of cases it turned out that the earlier conclusions had been based upon insufficient evidence. Hence teachers should be very cautious in reference to historical data. In fact, mathematical histories have been getting out of date so rapidly that it is not safe to place much confidence in some of the conclusions of the histories which are several years old. It is however, true that many important central facts of mathematical history rest on sound foundations and one can often see from the nature of the question and from the evidence presented whether any reasonable doubts remain.

In support of the statement that mathematical histories have been getting out of date very rapidly we may note the fact that in the third edition (1907) of volume I of Cantor's *Vorlesungen über Geschichte der Mathematik,* page 616, there occurs the following sentence: "Ist noch unserem Dafürhalten die Erfinding der Nul eine babylonische, die Vertiefung des Begriffes eine indische, so ist das Rechnen mit der Nul schon zu Brahmaguptas Zeit Gegenstand besonderer Vorschriften gewesen." The corresponding sentence in the second edition (1894), page 576, begins as follows: "Ist noch unserem Dafürhalten die Erfindung der Nul eine indische." The substitution in the newer edition of the word "babylonische" for the word "indische" is important.

The most popular history of mathematics in the English language seems to be Ball's *Short Account of the History of Mathematics,* which reached its fifth edition in 1912 and has been translated into Italian and also into French. That this popularity does not imply unusual accuracy may be seen from the review of the fourth edition, published by G. Eneström in the *Bibliotheca Mathematica,* volume 10 (1910), pages 85 to 88. Many interesting points relating to the early developments of mathematics in our own country are found in Cajori's *Teaching and History of Mathematics in the United States,* 1890. Cajori's *History of Elementary Mathematics* (1896) and *History of Mathematics* (1894), and the English translation of Fink's *Brief History of Mathematics* (1900) are also valuable works for those who read English only. The student of history of mathematics is compelled to examine journals devoted to such history, if he wants to be up to date. The best of these journals at the present time is the *Bibliotheca Mathematica.*[1]

The Teaching of Mathematics.—The most striking feature of the current mathematical literature is that such a large amount of it is devoted to teaching. This is doubtless largely due to the influence of the work which is being done under the direction of the International Commission on the Teaching of Mathematics. This commission was appointed in 1908 and it was able to present 156 of its publications at the recent international mathematical congress. The reports prepared by the twelve American committees have been published by the United States Bureau of Education and have been distributed freely. They constitute a valuable addition to the literature which should be read by all teachers of mathematics.

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[1] An example of valuable historical material appearing first in a journal is Professor Cajori's "*History of the Logarithmic and Exponential Concepts*" now being published by the MONTHLY, and not yet completed when this address was delivered.

Notwithstanding the great importance of the recent pedagogical literature we cannot pursue this subject further, since our aim has been to confine ourselves practically to mathematics as a branch of knowledge. In closing we may call attention to a type of literature which is often helpful to awaken interest where other means fail. I refer to the puzzles and curiosities which are apparently isolated from any important theory. A large collection of these is found, for instance, in the fifth edition (1911) of Ball's *Mathematical Recreations and Essays.*

BOOK REVIEWS.

The Teaching of Mathematics in Secondary Schools. By ARTHUR SCHULTZE. The Macmillan Company, New York, 1912. xx+370 pages. $1.25.

Instead of treating the general problems which face a teacher of mathematics in the high school, the author has chosen to provide a text which covers in detail the teaching of algebra and geometry. This makes the book of particular value to the inexperienced teacher.

With this end in view, the theoretical discussion has been cut down to the mere essentials. The present inefficiency of mathematical teaching, the reasons for retaining mathematics in our high school course, the different methods of teaching with their advantages and difficulties, the logical foundations of algebra and geometry, are rapidly reviewed. Such a brief summary must, of necessity, consist of little more than a statement of facts. An outline of this kind might be rendered more valuable both to old and new teachers by references to more detailed discussion of certain topics. Such references the author has in most cases failed to provide.

For the inexperienced teacher, the first book of plane geometry is full of difficulties. The author has attempted to smooth out some of these by a very detailed treatment. He discusses the characteristics of a good definition, and points out the difficulty in forming a definition. He also gives a number of examples to illustrate the preliminary definitions. The author would not have too great stress laid on the first few propositions. He would rather assume their proof and then impress their value by means of simple exercises.

The author insists that the original exercise is the important part of geometrical teaching, for it is only through it that the student gains power to think, and more than this, the original is more under the control of the teacher than the ordinary proposition. Methods of forming exercises are given and a large number of graded problems on the first propositions in triangles are added.

The discussion of the propositions of the first book centers about the equality of triangles and the properties of parallel lines. The treatment includes not only methods of proof but such incidental matters as converses of propositions, use of symbols and logical arrangement. Throughout the discussion in book one, the author uses the arrangement of propositions of Schultze and Sevenoak's Geometry. This renders the work less valuable to those teachers who use a text with radically different arrangement.

For the rest of geometry, the treatment is not so detailed. The general topics of the circle, similar figures, methods of attack for theorems and problems, short cuts for the teacher, are given. There is a chapter on limits which gives some very good suggestions in regard to teaching that subject.

For solid geometry, the treatment is confined to a discussion of the aim of the subject, the use of models, and methods of drawing figures. A chapter on applied problems, in which the author takes conservative ground, completes the treatment of geometry.

There are two points of criticism in regard to the discussion of geometry. Little emphasis has been put on the use of algebra in geometry. In view of the fact that algebra is taught before geometry, a more extended use of algebraic methods would seem wise. Again, the geometry course is necessarily limited in time and if more originals are taught, the text must give way. New teachers hardly know what to omit, and an outline of important propositions, if only for the first book, would be valuable, such, for instance, as is proposed by the National Committee of Fifteen on Geometry Syllabus.

The treatment of algebra is much more limited. After discussing what topics should be taught, the author takes up the methods of presenting the difficult ones, such as negative and irrational numbers. There is also a treatment of the use of graphs in the solution of problems, and a discussion of the use of logarithms. A chapter on the use of trigonometry by high school students completes the book.

Royal R. Shumway.

Higher Algebra for Colleges and Secondary Schools. By Charles Davison, Sc.D., Mathematical Master at King Edward's School, Birmingham. Cambridge: at the University Press, 1912. vi+320 pages. $2.00.

The introduction to this book is in part a review of some of the ground covered in the author's *Algebra for Secondary Schools*. We find here four chapters, treating: the binomial theorem for positive integral index, product and polynomial theorems, partial fractions, and complex quantity.

The first chapter contains chiefly examples and exercises which bring out important properties of the binomial coefficients, such as the following: (1) Show that $(n+1)_nC_0 - n_nC_1 + (n-1)_nC_2 - \cdots (-1)^n{}_nC_n = 0$; (2) Show that the integral part of $(a + \sqrt{a^2-1})^n$ is odd if n be a whole number. Here evidently a also should be a whole number.

The second and third chapters treat briefly in twelve pages the other subjects mentioned above, and include four lists of exercises. The latter are of a distinctly higher type than those found in American texts on College Algebra, yet are within reach of students above the freshman year. The same is true generally of the large number of interesting and instructive lists of exercises scattered through the text.

The treatment of complex quantity in Chapter IV is geometric, the vector OP being represented by one of the symbols (a, α) or (ρ, θ). Addition is based on the parallelogram, which includes the construction for the sum of real numbers.

"To multiply one complex number (a, α) by another (b, β), the number (a, α) must be treated in the same way that unity is treated to obtain (b, β)." Following this out geometrically gives $(a, \alpha) \times (b, \beta) = (ab, \alpha + \beta)$. Then i turns out to be the unit vector at right angles to the initial line, and this leads to $(\rho, \theta) = x+yi$. The chapter closes with a discussion of the commutative and associative laws, and of the nth roots of unity, followed by a list of exercises.

Part II contains six chapters dealing with convergency and divergency of series, binomial theorem, any index, exponential series, logarithmic series, summation of series, recurring series.

While the selection of the material here is excellent and well adapted to interest the student, the treatment is rather loose in places. On page 38 the sum of the geometric series is given as the sum of its first n terms. Several errors are due to statements about series in general which are true only for series of positive terms. Thus on page 42: "The ratio $u_1 + u_2 + u_3 + \cdots : v_1 + v_2 + v_3 + \cdots$ lies between the least and greatest of the ratios $u_1 : v_1, u_2 : v_2, u_3 : v_3, \cdots$." Nothing is said about the signs of the u's and v's. The same statement occurs on page 121.

The treatment of the binomial theorem, any index, in Chapter VI extends only to rational indices. This chapter also contains Vandermonde's theorem, two theorems on homogeneous products, and a considerable number of exercises.

In Chapter VII on exponential series, e is defined by means of the usual series, and is then computed to four places by using the first ten terms. It is not shown that the omitted terms could not affect the fourth place. Then $f(m)$ is defined as the exponential series in m, and it is shown that $f(m) \cdot f(n) = f(m + n)$, and from this that $f(x) = e^x$ for rational values of x. Other values of x are not considered.

The logarithmic series in Chapter VIII is derived by aid of the binomial theorem and hence is obtained only for rational values of x. In computing log 2 to four places, it is again not shown that the remainder of the series could not affect the fourth place. A number of interesting exercises on logarithms are given; the last one calls for evaluating the limit of the expression which leads to Euler's constant.

Under summation of series in Chapter IX we find the sums of some special series, definitions of the polygonal and figurate numbers, and the method of summation by differences, including numerous worked examples and exercises. Recurring series are briefly treated in Chapter X.

Part III contains four chapters, treating of inequalities and maxima and minima, approximations, limits and differential coefficients, convergency and divergency of series (second treatment).

In the first of these chapters occur a number of "important inequalities," several of which require restrictions that are not stated. Thus we find: "If $a > b$ then $a^m > b^m$ when m is a positive integer." Again: "If $a > x$, $b > y$, $c > z$, then $abc > xyz$." In the work on inequalities which follows, more care is taken to make the statements accurate. We have here comparisons of the arithmetic and geometric means of n quantities, and of the mean of the mth

powers with the mth power of the mean of n quantities. Under maxima and minima we find for example: "The maximum of $x^a y^b z^c$, when $x + y + z = C$."

The chapter on approximations consists chiefly of many instructive examples and exercises. In the next chapter we find the usual theorems on limits, with the usual intuitive proofs. This is followed by a treatment of some indeterminate forms. Then the derivative is defined and polynomials and products of linear factors are differentiated. The last exercise of the chapter rather strains the use of the equality sign.

Chapter XIV concludes the treatment of convergency and divergency of series. Additional tests are given, a direct comparison test, Raabe's test, and a logarithmic test. In the proofs of all of these, the necessary restrictions on the signs of the terms are not stated.

Part IV contains eight chapters treating the relations between coefficients and roots, transformation of equations, properties of equations, equal and commensurable roots, cubic and biquadratic equations, location of roots (Sturm's theorem), Horner's method, and determinants. The presentation of this material is clear and direct, and is accompanied by numerous well-chosen examples and exercises.

One or two slips should be noted. The theorem on the number of roots, page 152, states more than is proved, since the existence of a root is not established. On page 177, under Rolle's Theorem, is the rather remarkable statement: "It follows that every root of the equation $f'(x) = 0$ lies between a pair of successive roots of the equation $f(x) = 0$." Similarly on page 185 under equal roots we find: "If the roots of $f(x) = 0$ be all different, all the roots of the equation $f'(x) = 0$ lie between successive pairs of roots of the given equation."

Chapter XXII develops the principal properties of determinants, the elements of a determinant being obtained by permutation of the subscripts in the principal term. Application is made to solution of equations and to elimination, the latter including Bezout's method.

Part V contains five chapters relating to continued fractions, recurring continued fractions, continued fractions and series, theory of numbers and indeterminate equations. In each case a few elementary theorems are demonstrated, and these are followed by numerous worked examples and lists of exercises.

This concludes the body of the text, 279 pages. The next thirty pages contain twenty topics for short essays and fifteen "problem papers." The book closes with a list of answers.

Aside from such defects as those noted, the presentation of the material in the book is excellent. Perhaps two thirds of its pages are devoted to worked examples and exercises, and these are of such variety that they would make an interesting collection for any teacher of college mathematics.

The book seems to be quite free from typographical errors. Only three were noticed: on page viii, line 3, the radical sign should be deleted; on page 140, line 7, a transposition is needed; on page 179, line 7, a fraction line is missing. To the list of symbols should be added the following: Σ, $\sim$, $\nless$, and $\ngtr$.

W. C. BRENKE.

PROBLEMS AND QUESTIONS.

The solutions of problems are omitted this month to give more time for reports on problems proposed in the January issue. The following generalized puzzle is in keeping with the generalizing spirit of pure mathematics. It was suggested at a dinner of the Chicago Section of the American Mathematical Society.

A Puzzle Generalized.

By R. P. BAKER, University of Iowa.

The original form of the puzzle is as follows: Given 6 counters in two like sets of 3 arranged alternately in a line. By moving pairs along the line without rotation to arrange in 3 moves the like sets together, leaving no gaps.

By using $+$, $-$ for the counters and 0's for the gaps the solution may be indicated thus.

$$\begin{array}{cccccccccc}
+ & - & + & - & + & - & & & & \\
+ & - & + & 0 & 0 & - & - & + & & \\
0 & 0 & + & + & - & - & - & + & & \\
 & & 0 & 0 & - & - & - & + & + & +
\end{array}$$

The generalized problem substitutes m for 3 and $2m$ for 6, m being any integer.

The first proposition is that less than m moves is impossible. At first we have $2m - 1$ changes of sign and at the last 1. Hence $2m - 2$ changes are lost. Now no move can destroy more than 2 changes. For consider the moved pair and its former and latter neighbors. If the moved pair is like, not more than 2 changes exist at the start and so no more can be lost. If the moved pair is unlike, 3 changes may exist at the start but one is carried over. The first move cannot destroy more than one change and so m moves are necessary.

The solution is, however, for $m > 3$ not unique in its details, but can be described in its necessary features, and the latitude permitted in the way of variants indicated. The plan differs according to the residue of m modulo 4.

For $m = 4k$, arrange to count $2m + 2$ columns, the last two being occupied by zeros at the start, and for convenience cut off the last m columns by a bar

$$\begin{array}{ccccc|ccccc}
1 & 2 & 3 & \cdots & m+2 & m+3, & m+4, & \cdots\ 2m, & 2m+1, & 2m+2 \\
+ & - & + & & - & + & - & - & 0 & 0
\end{array}$$

We describe a move by a substitution symbol. Thus $\begin{pmatrix} a, b \\ \alpha, \beta \end{pmatrix}$ means that the counters in the columns a, b are moved to the columns α, β.

The scheme of moves, in the first stage is now,

$$\begin{pmatrix} 2 & 3 \\ 2m+1, & 2m+2 \end{pmatrix}\begin{pmatrix} m+5, & m+6 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 6 & 7 \\ m+5, & m+6 \end{pmatrix}\begin{pmatrix} m+9, & m+10 \\ 6 & 7 \end{pmatrix}\cdots$$

$$\begin{pmatrix} 2m-3, & 2m-2 \\ m-6, & m-5 \end{pmatrix}\begin{pmatrix} m-2, & m-1 \\ 2m-3, & 2m-2 \end{pmatrix}\begin{pmatrix} m+1, & m+2 \\ m-2, & m-1 \end{pmatrix}.$$

The numbers on the left of the bar, are now

$$1,\ m+5,\ m+6,\ 4,\ 5,\ m+9,\ m+10,\ \cdots\ m+1,\ m+2,\ m,\ 0,\ 0.$$

Since odd numbers are indicated by $+$ and the even ones by $-$, they are seen to occur in like pairs. The numbers to the right are

$$m+3, m+4, 6, 7, m+7, m+8, 10, 11, \cdots m-2, m-1, 2m-1, 2m, 2, 3.$$

They begin and end with a $+$, the interior being occupied with alternate like pairs.

The number of moves taken so far is seen, by considering the successive pairs, to be

$$\frac{2(m-2-2)}{4}+2=\frac{m}{2}=2k.$$

There are now k pairs of $+$'s and k pairs of $-$'s on the left, and on the right $k-1$ pairs of $+$'s with k pairs of $-$'s, the extremes being also single $+$'s.

The second stage proceeds by moving a pair of $-$'s from right to left, and a pair of $+$'s from left to right to replace them, the only restriction on order being that the pair of $+$'s in the first two columns $(1, m+5)$ is to be moved last. This avoids breaking the rule concerning gaps. In this stage $2k$ moves are made, and in all $4k = m$ moves.

For $m = 4k+1$, the first part comprises $2k+1$ moves, the second $2k$ moves.

For $m = 4k+2$, there are $2k+1$ moves in each part.

For $m = 4k+3$, there are $2k+1$ moves in the first part, $2k+2$ in the second.

To save the lengthy symbolic representation it is sufficient to give the solutions for $m = 5, 6, 7$ as far as the end of the first stage.

If $m = 1$ the problem is already solved, for $m = 2$ it is impossible, and for $m = 3$ the solution does not conform to the general scheme; the reason being the lack of partners to carry over the marker which is left single at the end of the line.

$$m = 5$$

$$\begin{array}{l} \left\{\begin{array}{ccccccc|ccccc} + & - & + & - & + & - & + & - & + & - & 0 & 0 \\ + & 0 & 0 & - & + & - & + & - & + & - & - & + \\ + & + & - & - & + & - & 0 & 0 & + & - & - & + \end{array}\right. \\ \phantom{\{}\begin{array}{ccccccc|ccccc} + & + & - & 0 & 0 & - & - & + & + & - & - & + \end{array} \end{array}$$

$$m = 6$$

$$\begin{array}{l|l}
\left\{\begin{array}{cccccccc} + & - & + & - & + & - & + & - \\ + & 0 & 0 & - & + & - & + & - \end{array}\right. & \begin{array}{cccccc} + & - & + & - & 0 & 0 \\ + & - & + & - & - & + \end{array} \\
\left\{\begin{array}{cccccccc} + & + & - & - & 0 & 0 & + & - \\ + & + & - & - & - & + & + & - \end{array}\right. & \begin{array}{cccccc} + & - & + & - & - & + \\ + & 0 & 0 & - & - & + \end{array}
\end{array}$$

$$m = 7$$

$$\begin{array}{l|l}
\left\{\begin{array}{ccccccccc} + & - & + & - & + & - & + & - & + \\ + & 0 & 0 & - & + & - & + & - & + \\ + & + & - & - & + & - & + & - & 0 \end{array}\right. & \begin{array}{ccccccc} - & + & - & + & - & 0 & 0 \\ - & + & - & + & - & - & + \\ 0 & + & - & + & - & - & + \end{array} \\
\left\{\begin{array}{ccccccccc} + & + & - & - & + & 0 & 0 & - & - \\ + & + & - & - & + & + & - & - & - \end{array}\right. & \begin{array}{ccccccc} + & + & - & + & - & - & + \\ + & 0 & 0 & + & - & - & + \end{array}
\end{array}$$

for higher numbers the extra moves occur between the braces.

NOTES AND NEWS.

The Royal Society has awarded the Copley medal to Professor Felix Klein, of Göttingen, for his researches in mathematics.

A second, enlarged, edition of Professor W. F. Osgood's *Lehrbuch der Funktionentheorie* has appeared from the press of B. G. Teubner, of Leipzig.

Henry Holt and Company have just brought out a text on the *Theory and Practice of Mechanics* by S. E. Slocum, professor of applied mathematics in the University of Cincinnati.

Readers interested in studies of fundamental assumptions will enjoy a paper by A. B. Frizell on a set of 80 "Axioms of Ordinal Magnitudes," presented last August at the International Congress of Mathematicians.

A Peruvian knot record is described by Mr. L. Leland Locke in the *American Anthropologist*, April–June, 1912, p. 325. Apparently the knotted cords were used simply as numerical records. They seem to constitute "the earliest known decimal notation of the Western World, at any rate the earliest that admitted of easy transportation."

Professor Frederick Anderegg, of Oberlin College, will be absent on his sabbatical year during 1913–14. He expects to spend the time in Europe.

The Cambridge University Press has published a pamphlet of 51 pages by Sir Thomas L. Heath on "The Method of Archimedes," which was discovered by Heiberg in 1906 in Constantinople. The Greek manuscript in question is a tenth century copy of the works of Archimedes. Later the original writing was partially washed out and the parchment used for liturgical writing. In most places the original script is still legible. Among other works of Archimedes this manuscript contains "The Method" which had been thought irretrievably lost. It indicates the steps by which he worked his way to mathematical discoveries.

Professor F. Klein announced in the October–December, 1912, number of the *Jahresbericht der Deutschen Mathematiker-Vereinigung* that the mathematical volume, of the series of about 80 volumes which are being published by B. G. Teubner under the general title "Die Kultur der Gegenwart," will contain six monographs under the following headings: The relations between mathematics and general civilization; mathematics and philosophy; the mathematics of antiquity and the middle ages; the mathematics of the sixteenth, seventeenth and eighteenth centuries; the mathematics of recent times; the teaching of mathematics. The third of these monographs has appeared and the first and second are to appear before very long.

Professor W. D. Cairns, of the department of mathematics of Oberlin College, is offering this year, as last, a so-called outline course in mathematics, designed for those students who do not purpose to continue the study beyond the requirements of the freshman year. Beginning with the usual work in trigonometry as a prerequisite, this course gives the students a limited, but a definite, acquaintance with the essentials of plane analytic geometry and graphical methods, followed by a similar treatment of calculus methods, with applications throughout to geometry and mechanics, but more particularly to analytic and graphic treatment of economics, biology, etc. It is hoped that in this way those who major in the lines just mentioned may the more readily understand the mathematical matter which now forms so important a part of the development of these subjects. It serves also as an "informational course" for the general student who will never need to use the methods, but will wish to obtain thus a survey, exact as far as it goes, of this particular field.

The Library of Congress has published index cards of the parts of the German and French Encyclopedias which have appeared. Other cards will be issued as new numbers are published. These cards index not only the separate articles under their various authors but they contain also one or more general English terms indicative of the contents of these articles. For instance the card headed "Borel, Recherches contemporaines sur la théorie des fonctions," closes with the following sub-headings in English: (1) Functions. (2) Aggregates. (3) Integrals. (4) Series, Infinite. These cards can therefore be easily arranged in the form of a subject catalogue, and the Library of Congress furnishes duplicate cards for this purpose, at a reduced price. In view of the slow rate at which these encyclopedias are being published, it would be very useful to have index cards entering into greater details. The present cards will however do much to simplify the use of these standard works of reference, and the mathematicians are greatly indebted to Brown University for furnishing copies to the Library of Congress.

Errata in the February issue. On page 44, equation (1) should read $i\varphi = \log(i \sin \varphi + \cos \varphi)$. Likewise each ψ on this page should be replaced by φ. Also in line 16 from the bottom $e^{+r\sqrt{-1}}$ and $e^{+r\sqrt{-1}}$ should read $e^{+v\sqrt{-1}}$ and $e^{-v\sqrt{-1}}$.

On page 59, the names in line 7 from the bottom should be E. L. Bates and F. Charlesworth.

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ORGANIZATION OF THE EDITORIAL BOARD

The Board of Editors of the MONTHLY is organized under Articles of Agreement in accordance with which the officers and the editorial and executive committees are elected by majority vote of all the members, the other committees being selected by the managing editor.

The names of the officers and committee members were given in the January issue in the various departments, and they are here collected for convenient reference.

The members of these committees will welcome suggestions and criticisms from all sources, especially from those who are interested in helping the MONTHLY to realize its ideals.

The need of a standard journal in this country, with aims such as those of the AMERICAN MATHEMATICAL MONTHLY, will not be questioned. But its ultimate success will depend upon its speedily becoming self-supporting. This can happen only if the individuals who should form its constituency become subscribers in large numbers. The subscription list is rapidly increasing. All friends of the cause can assist in the good work by passing the word along to others and by sending to the Managing Editor the names of those who should be interested in such a journal. School libraries and public libraries should be included in the list.

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SOME THINGS WE WISH TO KNOW

1. **Dangers.** It may be well to mention first some of the things we do not wish to know. Broadly speaking, these fall under two heads: (*a*) purely personal opinions not supported by facts or experiments; (*b*) exhaustive treatments of the entire field.

Topics that are too broad in their scope are attractive to the writer on pedagogical material because they admit of broad generalities and have the allurement of apparently larger accomplishment. This is just the reason why we do not care to encourage them, for it is evident that a broad topic, such as *The value of mathematics*, or *How algebra should be taught*, not only allows glittering generalities of statement, but is likely to contain very little of specific worth on any one given point.

Broad topics also savor strongly of the kind of *ex cathedra* statements of personal opinion mentioned above. A statement that a given topic is (or is not) vital, or interesting, or valuable, or good for the student, without supporting reasons or statistics, is very common, especially in articles on mathematics. This practice we cannot too strongly condemn and we intend to discourage it to the limit of our ability; not that we would contend that a given topic is *not* interesting, for on the contrary, we are inclined to believe that a great variety of topics are extremely interesting. But that is not getting very far in deciding whether the topic should be taught, and how.

2. **The Basis of Knowledge.** It seems to demand an apology if we state to readers who are scientists that the basis of knowledge not received by revelation is either experimentation or deduction. As a defense for such a truism, we need only point to the many papers that show evidence of some other method of reaching conclusions.

In the Dark Ages, and for a period thereafter, medical practice included the administration of blood-letting; and it was often asserted that this process was beneficial to the patient. They who first arose to demand statistics upon this and other hallowed practices were naturally regarded as heretics, and they were treated with due scorn and derision.

May we very hesitatingly ask for a few statistics regarding some of our own hallowed traditions? And do we thereby incur displeasure? Shall it be said of us that we are attempting the overthrow of mathematics? In any event, that is what we mean by *knowing*, for we are aware of no other basis of knowledge than such experimentation as we here imply.

3. **Several Topics.** Among fair topics for such a really valuable discussion, which we would hold to be quite open to both opinions, the usual process for discovering the rational roots of an ordinary algebraic equation seems to be a good illustration.

In how far is this topic a desirable one? We know that it is theoretically always effective. But in how far is it really called for? In how far do students that have been taught the process retain it? How long do they retain it? Do they actually use it? Is the difference between groups of students (not individuals) that have been taught the process and those who have not, traceable in any manner after one year? What difference, if any, manifests itself, in such groups, either in knowledge or in behavior?

The consideration of the possible value of having learned, even a forgotten topic, is here by no means ruled out of consideration. How does this value manifest itself? If it is real, it should be traceable in a *group*. But is this the *only* claim which is valid for this particular topic?

These things we wish to know. They can be found out. Nor is the topic too small to warrant a critical examination and report. We most earnestly assert that the records of real work on this topic would be more welcome and more valuable than an extensive expression of mere opinions on the whole of algebra.

The amount of work necessary to really contribute to such a topic becomes apparent upon short consideration. There are other things we wish to know that can be settled quickly. For example, we would be glad to know what percentage of men entering various universities know how to solve a numerical quadratic; what percentage know how to solve a linear equation; what percentage know how to add, subtract, multiply and divide with simple fractions such as 3/5 and 4/7. Here again mere expressions of opinion as to what should be the case are not desired. We would be glad to know how these percentages change every six months for two years. Does the training received in mathematical courses perceptibly affect the ability to manipulate simple fractions? Safeguards against erroneous conclusions are to be found in books and papers on psychology.

4. **A Question about Tables.** Another question that is in serious need of discussion is the most efficient use of types and spacings in tabular matter of all sorts. The variety of types in actual use suggests the desirability of some accurate study, and the mention of psychology suggests it. To be sure, experiments have already been made. These, though valuable, are not complete nor exhaustive. What has been done is quite unknown, apparently, among mathematicians. Here again, expression of personal preference is not at all trustworthy or valuable. Nor is the persistence of an antiquated form of type in tabular matter an argument in its favor, as has been pointed out frequently in reference to the German

"Gothic" characters. That the spacing into groups of threes or into groups of fives laterally and vertically has an important bearing upon legibility and quick understanding is highly probable; that the use of white spaces in the place of black rulings works great change is undoubted. Which of these, and what other features are to be preferred in tables, is of great consequence. These questions can only be answered after very exhaustive experiments of no mean order. A competent psychologist and a competent mathematician working jointly would make the best combination for attempting such a great task.

5. **Forthcoming Articles.** We are glad to announce that short articles dealing with very specific topics are already promised. Among others we may mention: an article on the teaching of solid angles; one on the desirability of accurate space drawings; one on methods of interpolation in a table of logarithms of the trigonometric functions for small angles; one on the elements of the theory of insurance. It is hoped that these and others equally definite in topic will not be long delayed. In this connection see the note at the bottom of page 140 of this issue.

E. R. Hedrick, *for the editors.*

HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

By FLORIAN CAJORI, Colorado College.

IV. From Euler to Wessel and Argand. 1749–about 1800.

A Half Century of Barren Discussion on Logarithms.

In an essay *Sur les logarithmes des quantités negatives*[1] D'Alembert refers to his correspondence of 1747 and 1748 with Euler on this subject and states that he had read Euler's paper of 1749, but felt still that the question was not settled. D'Alembert proceeds to advance arguments of metaphysical, analytical and geometrical nature which shrouded the subject into denser haze and helped to prolong the controversy to the end of the century. Defining logarithms by the aid of two progressions, as did Napier, D'Alembert asserts that the logarithms of negative numbers are not imaginary, but real, or rather, that they may be represented at will as real or imaginary, since everything hinges on the choice of the system of logarithms. As his metaphysical argument against Euler's conclusion he gives it as improbable that the logarithmic curve $y = e^x$ passes from $x = \infty$ to imaginary values. As a geometric reason he asserts that all curves for which $a^n dy/y^n = dx$, n being odd, have two branches which are symmetrical with respect to the X-axis and yielding two values, y and $-y$ for one and the same value of x. He claims his proof to be general, hence true for $n = 1$.

[1] *Opuscules mathématiques*, T. I, Paris, 1761, pp. 180–209.

This conclusion, says Karsten,[1] rests upon an error in sign, so simple that he doubts whether D'Alembert was really in earnest in his argument. The disagreement between D'Alembert and Karsten was due to a lack of a clear and consistent definition of negative areas, and to the use of infinite areas involving the indeterminate form $\infty - \infty$. No doubt D'Alembert and others had a right to assume two branches to the logarithmic curve, if they wished; the whole question is one of assumption. If we agree that $y = e^x$ shall be so interpreted that for every fractional value of x with an even denominator, both real roots shall be taken, then there are two branches, of course; if we agree to restrict ourselves to the positive root only, then there is naturally only one branch. But what most mathematicians of that time failed to understand, was Euler's remark that the question of one branch or two branches has in fact nothing to do with the question as to the nature of logarithms of negative numbers. The former is a question in evolution, the second is a question in logarithmation; the former has reference to one of the two inversions of involution, the latter to the other inversion. In his paper of 1749 Euler had explicitly confined himself to the system with the positive base $e = 2.718 \cdots$ and had concluded that the logarithms of negative numbers are complex. D'Alembert selects a negative base by taking a geometric series $1, -2, 4, -8, \cdots$ and the arithmetic series $0, 1, 2, 3, \cdots$. Thus he obtains certain negative numbers that have real logarithms. When Euler says that $l(+a) = l(-a)$ yields zero for the logarithm of all complex numbers of unit modulus, D'Alembert in reply again changes his system of logarithms to one in which the arithmetic series is $0, 0, 0, \cdots$ and reminds the reader that any geometric series whatever can be associated with this. For one and the same system of logarithms D'Alembert selects the two series

$$\begin{array}{ccccccccccccccc} \cdots & 2n & n & 0 & -n & -2n & \cdots & \infty & \cdots & -2n & -n & 0 & n & 2n & \cdots \\ \cdots & -b^2 & -b & -1 & 1/b & 1/b^2 & \cdots & 0 & \cdots & 1/b^2 & 1/b & 1 & b & b^2 & \cdots \end{array}$$

in which the base is arbitrarily altered in passing from -1 to $1/b$; this is done to show that $lb^m = l(-b^m)$. What other mathematician of renown ever changed his base so often in defence of a mathematical theory!

Though not published until 1761, this paper of D'Alembert was written some years previous, certainly before the publication in 1759 of a paper, *Reflexions sur les quantitêes imaginaires*[2] by Daviet de Foncenex. This paper is the more interesting because its author, a pupil of Lagrange, remarks that Lagrange had communicated to him his views on the logarithms of negative numbers. Lagrange himself never published anything on this subject. De Foncenex gives an elementary proof for $\log(\cos\varphi + i\sin\varphi) = \varphi\sqrt{-1}$, then writes $\cos\varphi + \sqrt{-1}\sin\varphi = a + b\sqrt{-1}$, and takes for φ, $\varphi + 2\lambda\pi$. Then $(\varphi + 2\lambda\pi)\sqrt{-1} = \log(a + b\sqrt{-1})$. This cotesian result is derived by de Foncenex with the aid

[1] W. J. G. Karsten, "Von den Logarithmen verneinter Grössen," 1. und 2. Abtheilung, *Abh. Münch Acad.*, V, 1768, S. 55.

[2] *Miscellanea Philosophico-mathematica societatis privatœ Taurinensis*, I, 1759, pp. 113–146.

of the hyperbola, the very curve by which J. Bernoulli I derived results contradictory to this. He says that Bernoulli did not pay proper attention to questions of continuity. In $dz/z = -du/u$, the element of surface dz/z becomes finite while u is passing from $+\infty$ to $-\infty$. He argues that, since the element of surface changes its sign without passing through zero, there is no continuous transition from positive to negative surfaces of the hyperbola; since $z = a^u$ yields $du = dz/z$, there is no continuous passage from the + to the − branch of the logarithmic curve; the branches are real, but isolated and algebraically independent of each other, though transcendentally connected. Apparently in contradiction of his own earlier analytical conclusions, de Foncenex now grants that the logarithms of negative numbers may be considered real, like those of positive numbers.

A reply to de Foncenex was prepared by D'Alembert, under the title, *Supplément au mémoire précédent.*[1] This reply appeared at the same time and in the same volume as his first article. D'Alembert discusses questions of geometry, corrects some mistakes of de Foncenex and clings to some erroneous views of his own.

De Foncenex prepared a reply to D'Alembert's two papers,[2] in which he declares that the arithmetical side of this controversy must be kept distinct from the geometrical. If in $y = e^x$, e is a fixed number, there can be no real logarithms of negative numbers. In the study of the logarithmic curve, we must inquire into the existence of two branches and their relation to each other without using processes of integration. He finds the ordinate β of the evolute of $y = e^x$ to be $(2y^2 + 1) \div y$. Not satisfied with this, he puts $1 + y^2 = u$ and finds $\beta^2 = (4u - 4u^2 - 1) \div (1 - u)$. Hence, he says, there are two values for β or there are two branches of the evolute, both symmetrical with respect to the X-axis. Since the evolute has two branches, the curve itself must have two branches. He now admits D'Alembert's contention that the two branches are algebraically connected. He enters upon the impossible task of harmonizing results by writing $\varphi\sqrt{-1} = \log(\pm\cos\varphi \pm \sin\varphi\sqrt{-1})$, with the direction that the proper signs be chosen in every special case. It is seen that de Foncenex started out in close alignment with the views of Euler, but departed from them considerably through his study of questions of continuity of curves and his false interpretation of the roots of a quadratic equation.

D'Alembert wrote the article "Logarithme" for Diderot's *Encyclopédie ou dictionnaire raisonné*, 9, Paris, 1765. This article was later embodied in the great *Encyclopédie méthodique*, 1785, the mathematical part of which was translated into Italian in 1800. On the subject of logarithms of negative numbers the article voiced the views of J. Bernoulli I. rather than those of Euler. The only other Frenchman who before the closing years of the eighteenth century is known to us as participating in this discussion was Louis Charles Trincano, who con-

[1] *Opuscules mathématiques*, T. I, Paris, 1761, pp. 210–230.

[2] De Foncenex, "Eclaircissemens, etc.," *Mélanges de phil. et de math. de la soc. roy. de Turin pour les années* 1760–1761, pp. 337–344.

tributed a *Mémoire sur les logarithmes des quantités négatives* as an appendix to his father's *Traité complet d'arithmétique,* Paris et Versailles, 1781. The young man says truthfully enough that if one assumes the geometric progression $-2, -4, -8, \cdots$ or $2, -4, 8, \cdots$ or $-2, 4, -8, \cdots$ then some or all negative numbers have real logarithms. According to the first progression, positive numbers would have no real logarithms. In $2, 4, 8, \cdots$ the logarithm of a negative number is *chimérique.*

Notwithstanding the influence of D'Alembert, Euler's theory acquired a firm foothold in France at the close of the century. In 1797–9 appeared in Paris Lacroix's great treatise on the calculus, which presents logarithms unreservedly from the eulerian stand-point.[1]

It is worthy of note that we have not encountered a single research, published in England in this century and touching the topic under discussion. Can this be due to the alienation between English and continental mathematicians which resulted from the Newton-Leibniz controversy on the invention of the calculus?

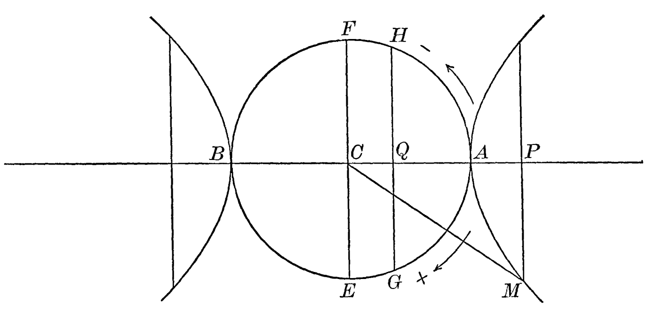
In Germany there appeared between 1750 and 1770 three writers who accepted the eulerian view and were unusually clear in their expositions. We refer to Charles Walmesley, an English Roman catholic prelate who was elected member of the Berlin Academy and contributed a paper to its memoirs, J. A. Segner, professor at Halle, and W. J. G. Karsten, professor at Bützow, later at Halle. Walmesley wrote an article of four pages, *Méthode de trouver les logarithmes de chaque nombre positif, négatif, ou meme impossible,*[2] which contains a brief derivation, by the integral calculus, of the formula $ac\sqrt{-1} = \pm \log(y \pm u\sqrt{-1})$, where ac is an arc of unit-circle, y and u the cosine and sine of the arc. From this formula, and the consideration of the periodicity of trigonometric functions, Walmesley deduces the Eulerian values for $\log 1$, $\log -1$, $\log \pm \sqrt{-1}$, $\log (a + b\sqrt{-1})$. In this article, as in Euler's article of 1749, y and $u\sqrt{-1}$ are tacitly taken along lines perpendicular to each other. To find the logarithms of $a + b\sqrt{-1}$, Walmesley changes it to the form $(a^2 + b^2)^{-\frac{1}{2}}(y + u\sqrt{-1})$; now he says, *il est clair qu'on doit prendre l'arc de cercle dont le sinus est u, le cosinus y,* then if the arc is φ and $\log(a^2 + b^2)^{\frac{1}{2}} = C$, we have for $\log(a + b\sqrt{-1})$ the values $C + \varphi\sqrt{-1}$, $C + (\varphi \pm 2\pi)\sqrt{-1}, \cdots$. We suspect that the diagram attributed to Wessel and Argand was in the *minds* of mathematicians long before it was recorded on *paper.* Walmesley closes with some general remarks disclosing an uncomfortable feeling toward the negative and imaginary. He declares it improper to speak of the ratio between a positive and a negative quantity, since ratio involves only magnitude, not quality. To mix the two ideas is like considering density, weight, etc., in comparing the volumes of two solids.

Segner lays unusual emphasis for that time upon the treatment of imaginaries and is the first to embody Euler's logarithms of complex numbers in a regular school text. In Segner's *Cursus mathematici, Pars* IV, *Halæ Magdeburgicæ,*

[1] S. F. Lacroix, *Traité du calcul dif. et calcul int.,* seconde éd., T. I, 1810, Introduction, §§81, 82.
[2] *Histoire d. l'acad. roy. d. sc. et belles-let.* [*Mémoires*], année 1755, Berlin, 1757, pp. 397–400.

1763, there is a discussion of the roots of unity, treated by De Moivre's formula, followed by a full treatment of the logarithms of complex numbers. Euler's position is here taken fearlessly. The subject is emphasized in *Pars* V (1768) in a section on *De usu imaginariorum*. Segner's treatment is analytical throughout and free from entangling alliance with the then puzzling questions relating to the logarithmic curve and hyperbola.

Karsten published an extensive article of 108 pages[1] in which D'Alembert's errors are brought out with great clearness. But Karsten agrees with D'Alembert in one point, namely that Euler's proof (quoted by us in full on p. 81) of the fundamental theorem, that $\log x$ has an infinity of distinct values, is objectionable in this respect: It conveys the idea that all logarithms belong to positive numbers, since $(1 + \omega)^n$, where ω is infinitesimal and n is infinite, appears to be always positive; the controversy itself relates to negative numbers. As a matter of fact, Euler makes no such restriction; he permits ω to be imaginary. His only error consisted in not giving an explanation sufficiently exhaustive for the needs of the time. Karsten is probably right in claiming that it would have been better, had Euler used in the proof a more elementary definition of a logarithm. Against D'Alembert, Karsten argues convincingly, that D'Alembert simply shows the possibility of systems in which negative numbers have real logarithms; the question at issue being, however, what are the logarithms of negative numbers in the natural system, with the base $e = 2.718 \cdots$? Most interesting in Karsten's paper is a geometric representation of the infinitely-multiple logarithms of a number. We have never seen a reference to this ingenious construction either by contemporaries of Karsten or by historians. It looks very much as if the transactions of academies had been in some cases the safest places for the concealment of scientific articles from the scientific public.



In going over Karsten's construction it must be remembered that he wrote 29 years before Wessel. In the hyperbola $x^2 - y^2 = 1$ and the circle $x^2 + z^2 = 1$, where Karsten takes $y = z\sqrt{-1}$, the "circle is an imaginary part of the hyperbola" and *vice versa*. An arc AG of the circle may be considered as an imaginary

[1] "Abhandlung von den Logarithmen verneinter Grössen," 1. und 2. Abtheilung, *Abh. Münch. Acad.*, V, 1768.

arc of the hyperbola. Each of the four arcs which come together at A or B is a continuation of each of the others, since all are expressed by a common equation. Between each two points M and G exist numberless different arcs: MAG, $MAGBFAG$, and in general, $MAG + 2\lambda\pi$, and also $MAHBG + 2\lambda\pi$, where $\lambda = 0, 1, 2, \cdots$. To any abscissa x there belong therefore not only numberless arcs, but also numberless corresponding sectors. Let one sector of the abscissa x be $\frac{1}{2}\log(x + y) = \sqrt{-1} \cdot \frac{1}{2}$ arc cos $x = \sqrt{-1}$ sect. cos x, the sector being assumed 0 when $x = 1$. If $x > 1$, say $x = CP$, then $\frac{1}{2}\log(x + y)$ gives the sectorial areas corresponding to the arcs $AM \pm \lambda(AGBFA)$, that is, $AM \pm 2\lambda\pi\sqrt{-1}$. Only the sector ACM is real. If $x < 1$, then $x + y$ is imaginary Let $x = CQ$, then $\frac{1}{2}\log(x + y)$ measures the sectorial areas whose arcs are $AG \pm 2\lambda\pi\sqrt{-1}$, or $AFBG \pm 2\lambda\pi\sqrt{-1}$. All these arcs and sectors are imaginary. From this we see that $\frac{1}{2}\log(-1)$ is represented by the imaginary sectors whose arcs are $AEB + \lambda(BFAEB)$, or also $AFB + \lambda(BEAFB)$. These values are all included in the formula $\log(-1) = \pm(2\lambda + 1)\pi\sqrt{-1}$. "Geometry is therefore so little antagonistic to the leibnizian tenet, that the logarithms of negative numbers are impossible, as actually to give it full confirmation."[1] In this representation of logarithms there is established a double correspondence: one between points in the plane, and x and y; the other between points in the plane, and x and z, where $y = z\sqrt{-1}$.

The country in which our controversy was carried on with greatest zest during the second half of the eighteenth century is Italy. We have already referred to de Foncenex's two papers. A few years later the question was taken up by the brothers Vicenzo Riccati and Giordano Riccati. They were sons of the celebrated Giacomo Francesco Riccati, of Venice, the originator of Riccati's differential equation. Vicenzo Riccati taught mathematics at the University of Bologna. In 1767 he addressed five letters to Jacopo Pellizzari, professor in Treviso, which, we are informed, were printed along with a letter of Giordano Riccati.[2] They were reprinted twelve years later.[3] The arguments of Leibniz, J. Bernoulli, D'Alembert, Euler and de Foncenex were discussed. As a whole, Riccati considers D'Alembert's criticisms of Euler valid, when Euler takes $1 + y/n = \cos(2\lambda - 1)/n\cdot\pi \pm \sqrt{-1}\sin(2\lambda - 1)/n\cdot\pi$, where y is the required $l(-1)$, and derives from it, for $n = \infty$, $l(-1) = \pm(2\lambda - 1)\pi\sqrt{-1}$, D'Alembert modifies the process by writing $\lambda = n$, $1 + y/n = \cos(2 - 1/n)\pi \pm \sqrt{-1}\sin(2 - 1/n)\pi$. Letting $1/n = 0$ in the right member, but not in the left, he gets $1 + y/n = 1$, or $y = 0 = l(-1)$. Riccati rejects D'Alembert's revision of Euler at this point, but sustains D'Alembert in criticizing Euler's $l(1 + \omega)^n = n\omega$, n infinite and ω infinitesimal, as embracing all the logarithms, since $(1 + \omega)^n$ must be a positive number (!), hence $n\omega$ does not by his analysis represent the logarithms of negative numbers. We have seen that Euler sets no limitation as to whether the infin-

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[1] Karsten, *op. cit.*, p. 103.
[2] P. Riccardi, *Biblioteca matematica Italiana*, Modena, 1893, Pt. I, Vol. II, 368.
[3] *Nuovo giornale de'letterati d'Italia*, Modena, 1779, T. XVI, pp. 137–219.

itesimal ω is real or imaginary and does not say that he means $(1 + \omega)^n$ to represent a positive number. As a matter of fact, Euler puts later $(1 + \omega)^n = x = -1$ and finds $l(-1)$ from this. Euler's lack of explicitness of statement at this point turned Riccati and others against him. Riccati writes $-1-\omega$ in place of $1+\omega$, thus making $l(-1-\omega) = \omega$, and arrives of course at $l(-1) = 0$. Moreover, the logarithmic curve was found to have two branches. The view of J. Bernoulli I and D'Alembert prevailed. Vicenzo Riccati wrote a letter to Joachim Pessuti, who had been in St. Petersburg and was a friend of Euler. Pessuti was then publisher of two literary journals, the *Anthologia romana* and *Effemeridi letterarie,* and after 1787 professor of applied mathematics at the Collegio della Sapienza in Rome. Pessuti defended the views of Euler in a paper, entitled *Riflessioni analitiche,* Livorno, 1777,[1] deriving Euler's main results from $u = \int \frac{dz}{1+z^2} = \frac{1}{2\sqrt{-1}} \log \frac{1+z\sqrt{-1}}{1-z\sqrt{-1}}$. Putting $z = 0$, he gets $u = n\pi$, where $n = 0, 1, 2, \cdots$. A reply to Pessuti made by V. Riccati appeared anonymously.[2] Riccati's attacks on Euler were repelled by a second writer who had come under the influence of St. Petersburg, the German astronomer Friedrich Theodor v. Schubert, in an article "Ueber die Logarithmen verneinter Grössen."[3] In the year 1778 Giuseppe Calandrelli, professor of mathematics at the Collegio Romano, attacked Pessuti and Euler in a pamphlet *Saggio analitico sopra la riduzione degli archi circolari ai logaritmi immaginarj,* Rome, 1778. As V. Riccati had done before him, so now Calandrelli casts doubts upon the correctness of the Bernoulli-Euler relation $\pi\sqrt{-1} = l(-1)$, the argument hinging on $2l(-1) = l1$. The same line of argument was pursued by P. M. Caldani, a pupil of V. Riccati and professor in Bologna. Caldani's *Della proporzione Bernoulliana fra il diametro, e la circonferenza del circolo e dei logaritmi,* Bologna, 1782, caused D'Alembert to pronounce Caldani the first mathematician in Italy.[4] Caldani objected, as Riccati had done before him, to Pessuti's writing $z\sqrt{-1} = 0$, when $z = 0$. Caldani writes $0 : 0\sqrt{-1} : : a : a\sqrt{-1}$; now, if $0 = 0\sqrt{-1}$, there follows $a = a\sqrt{-1}$, an impossibility. He wrote later a paper which we have not seen, *Riflessioni sopra un opuscolo del Padre Franceschini Barnabita, dei logaritmi de' numeri negativi,* Modena, 1791. These writers, including Pessuti, agree that Euler's proof of his fundamental theorem is not clear. When Euler writes $l(1+\omega) = \omega$, where ω is an infinitesimal, and then gets $l1 = 0$ when $\omega = 0$, he cannot at the same time claim, said the critics, that $\log 1 = 2\pi\sqrt{-1}, 4\pi\sqrt{-1}$, etc. Many of these points are discussed by the Spanish Jesuit Juan Andrès in a very interesting manner, in his book previously quoted. In 1766, when the

[1] Pessuti's paper is known to us only through the remarks on it found in Calandrelli's reply and in a book, *De studiis philosophicis et mathematicis,* Matriti, 1789, p. 215, written by JUAN ANDRÈS, and in FRÈRES (HOEFER), *Nouv. Biog. Générale.*

[2] *Nuovo giornale de'letterati d'Italia,* T. XV, Modena, 1778, pp. 144–204. See also T. XVIII, p. 107.

[3] Schubert's article is announced in the *Göttingische Anzeigen von gelehrten Sachen,* 1794, 2. Bd., p. 1313, as a reply to an article of Riccati in *Memorie della Soc. Italiana,* T. IV, p. 166.

[4] Frères, *op. cit.,* Art. *Caldani.*

Jesuits were driven out of Spain, he went to Italy and interested himself in the philosophic discussions there. He was inclined to accept Euler's conclusions, but thought that if $1 = e^0 = e^{2n\pi\sqrt{-1}}$, then we ought to have $0 = 2\pi\sqrt{-1} = 4\pi\sqrt{-1}$, etc.

Of higher type is the research of Gregorio Fontana, professor at the University of Pavia. He takes Euler's point of view. In an article, *Sopra i logaritmi delle quantità negativa e sopra gl'immaginarj*,[1] he tried to base the theory upon simpler and more rigorous proofs. He gives three proofs. First he takes $x =$ vers ϕ, and gets $\sqrt{-1}\,d\phi = \dfrac{dx}{\sqrt{x^2-2x}} = \left(\dfrac{xdx - dx}{\sqrt{x^2-2x}} + dx\right) \div (x - 1 + \sqrt{x^2-2x})$. Integrating, he gets $\phi\sqrt{-1} = \log\,(1 - x - \sqrt{x^2-2x})$ after letting $\phi = 0$ when $x = 0$. For $x = 2$, $\log\,(-1) = \pm(2n-1)\pi\sqrt{-1}$; for $x = 0$, $\log 1 = \pm 2n\pi\sqrt{-1}$. A second proof starts from $x = \cos\phi$ and involves integration. A similar third proof implicates several trigonometric functions. Fontana reasons with a clearness which is unusual in articles on logarithms of that day. He tells how at one time he thought he had absolute proof of the falsity of Euler's results. In $e^{\log(-a)} = -a$ he thought that $\log(-a)$ must surely be real whenever $-a$ is real. As a matter of fact, says he, we have $\log(-a) = \pm(2n-1)\pi\sqrt{-1} + \log a$. Finally Fontana derives the theorems without using the integral calculus; he uses infinite series, but pays no attention to questions of convergence.

This able article did not prevent the appearance of five "proofs" that $\log(-z) = \log z$, in Pietro Franchini's *Teoria dell'Analisi*, Rome, 1792. In 1795 Gianfrancesco Malfatti, of the University of Ferrara, discussed at length the question whether the logarithmic curve has one or two branches.[2] The equation $ydx = dy$ has an infinite number of integrals even under the restriction that $x = 0$ when $y = 1$. For any constant n, $e^{nx} = y^n$ gives rise to the differential equation $ydx = dy$. Before integrating one must agree upon the number of values y shall have. If it is to have one value, we get $e^x = y$; if two values, we get $e^{2x} = y^2$ or $(e^x - y)(e^x + y) = 0$. That $e^x = y$ has only one branch is argued from geometric and analytic considerations.

In regular mathematical text-books the lack of agreement relating to $\log(-1)$ was as marked as in the special articles. Odoardo Gherli [3]takes the ground that negative numbers differ from the positive simply in being taken in a contrary sense, and that logarithms take no cognizance of such contrarity. Pietro Paoli[4]

[1] *Memorie di matematica e fisica della società italiana*, T. I, Verona, 1782, pp. 183–202.

[2] *Memorie della reale accademia di scienze . . . di Mantova*, T. I, pp. 2–54. This article is known to us through a valuable historical monograph by Bernard F. Thibaut, *Dissertatio historiam controversiæ circa numerorum negativorum et impossibilium logarithmos sistens*, Göttingen, 1797, p. 20. Malfatti's article is reviewed in the *Göttingische Anzeigen von gelehrten Sachen*, 1796, 2. Bd., p. 1242. The *Göttingische Anzeigen* for 1786, 2. Bd., p. 1029, refers to another eighteenth century historical paper on this subject, namely the *De logarithmis numerorum negativorum* by Fridrich Mallet, professor at Upsala, which was printed in the *Nova Acta Reg. Soc. Scient. Upsaliensis*, Vol. IV, 1784, pp. 205–220.

[3] *Gli elementi teorico-pratici*, T. II, Modena, 1771, p. 297.

[4] *Elementi d'Algebra*, T. I, Pisa, 1794.

derives "il famoso Teoreme del Sig. Euler," but thinks he must use infinite series to secure rigor. Paolo Frisi, in his algebra,[3] shows familiarity with the leading writers on logarithms; he states the results reached by Euler, but is out of sympathy with them. Many results appear to him absurd, as for instance, $\log(-1) : \sqrt{-1} = \pi : 1$. He objects also to $0 \cdot \sqrt{-1} = 0$ and claims that an imaginary zero represents a real magnitude different from zero.

In a ponderous quarto volume,[2] published in 1782 at Florence, Petro Ferroni devotes a chapter to logarithms of negative numbers, only to arrive at the conclusion that analytic, as well as geometric, considerations of continuity yield $l(+1) = l(-1)$. Euler's argument, $-y = y(-1) = c^{ly} \cdot c^{\pm\lambda p\sqrt{-1}}$, hence $l(-y) = ly \pm \lambda p\sqrt{-1}$, is rejected, since by a purer doctrine (*puriore religione*) one has $-y = -c^{ly}$, and $l(-y)$ a real number. Ferroni bases his theory of general logarithms on the relation $y = \pm c^x$; he perceives no inconsistency in this procedure.

Signs of decadence are visible also in Germany. The high level reached there in the middle of the century was not maintained at its close. Kästner,[3] of the University of Göttingen, shows from the theory of ratios that we cannot have real logarithms of negative quantities. I. A. C. Michelsen, an admirer of Euler, a teacher at the royal gymnasium in Berlin, who in 1788 brought out a German translation of Euler's famous *Introductio* of 1748, added notes to that translation which indicate that Michelsen failed completely to grasp Euler's[4] argument; he holds to the view, for instance, that to every number, whether positive or negative, there corresponds one and only one logarithm. In 1795, G. S. Klügel[5] declared himself as follows: "The log $(-x^2)$ is impossible; but the log $(-x)$ is possible, else it must be shown that some condition does not permit x to be taken negatively." The possibility of the logarithm of a negative number rests upon the possibility of the negative number itself. Interest in the discussion of logarithms is shown in occasional notices that appeared in the *Göttingische Anzeigen von gelehrten Sachen*.[6] A notice of the year 1786 declares that the opponents of Euler tried to decide the question by formulæ of integration, radii of curvature, etc., thereby endeavoring to pass upon fundamental concepts by means of systems of notation, when, as a matter of fact, the concepts must be determined upon first, before the systems of notation are brought into action. Euler's opponents are dubbed calculators, rather than philosophers.

In conclusion, we stop to inquire, What was the result of the half-century of discussion?

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[1] *Paulli Frisii Operum tomus primus*, Mediolani, 1782, p. 188, etc.

[2] P. Ferroni, *Magnitudinum exponentialium logarithmorum et trigonometriæ sublimis theoria nova methodo pertractata*. Florentiæ, 1782, Chap. IX.

[3] *Leipziger Magazine f. reine u. angew. Math.*, Stück IV, 1786, p. 531. See also Cantor, *op. cit.*, IV, 1908, p. 313; B. F. Thibaut, *op. cit.*, p. 21.

[4] Euler, *Einleitung in d. Analysis d. Unendlichen*, 1788, p. 503.

[5] *Hindenburg Archiv*, 1795, 3. Heft, pp. 309–319; 4. Heft, pp. 470–481.

[6] See 2. Bd., 1786, p. 1029; 2. Bd., 1792, pp. 1185, 1186; 2. Bd., 1794, pp. 1029, 1313; 2. Bd., 1796, p. 1242.

The prevailing feeling toward the theory of logarithms of negative numbers was one of hesitancy. In 1799 Christian Kramp, professor of mathematics in Strassburg, expressed doubts[1] on the correctness of the formula $l(-x) = l(-1) + lx$.

In 1801 Robert Woodhouse, of Caius College, published a paper "On the Necessary Truth of Certain Conclusions obtained by Means of Imaginary Quantities." He says:[2] "The paradoxes and contradictions mutually alleged against each other, by mathematicians engaged in the controversy concerning the application of logarithms to negative and impossible quantities, may be employed as arguments against the use of those quantities in investigation."

In 1803 L. N. M. Carnot said in his *Géométrie de position*, p. III, that the long debate has failed to clear up the paradox that, while $\log(-2)^2 = \log(2)^2$, we cannot write $2\log(-2) = 2\log 2$. Euler had this point cleared up in 1749; he had the assumptions, under which $2\log(-2)$ is or is not equal to $2\log 2$, set forth in masterly fashion, but the discussion of fifty years involved the more primitive question, whether $\log x$ is or is not equal to $\log(-x)$. Why was Euler's brilliant paper of 1749 not convincing? For three reasons:

1. Euler failed to make plain that the question, whether the logarithmic curve had one branch or two branches, had in reality nothing to do with the theory of logarithms of negative and complex numbers.
2. Euler's proof of his fundamental theorem, that $\log x$ has an infinite number of values, was not explained with sufficient fulness of detail to carry general conviction.
3. The rank and file of mathematicians had not yet learned to treat with care the many-valued inverse operations and the vexing questions of discontinuity. Nor did they avoid the danger of confusion resulting from the simultaneous consideration of logarithms of different bases.

A noteworthy feature of the long discussion deserves mention. Ordinarily one genuine proof is accepted as sufficient to establish a mathematical truth with absolute certainty; no argument can be advanced against such a proof. But, as a rule, the discussion of logarithms of complex numbers was considered as involving arguments on each side of the question, so that the decision seemed to rest on a preponderance of arguments or a preponderance of probabilities, which apparently rendered it desirable for a partisan, in lawyer's fashion, to advance as many different "proofs" as possible. This attitude was particularly noticeable in the debates carried on by correspondence. It was due mainly to a lack of sharp definition of terms and to a failure to discriminate between what is assumed without proof and what is to be subjected to rigorous demonstration.

The Union of the Logarithmic and Exponential Concepts.

The wider adoption of the definition of logarithms as exponents, in school books, was due largely to the influence and example of L. Euler, who gave it in

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[1] Montferrier's *Dic. d. scienc. math.*, *T.* II, Paris, 1836, p. 185, Art. "Logarithme," where reference is made to Kramp's *Analyse des réfractions astronomiques et terrestres*, Strasbourg, 1799.

[2] *Philosoph. Transactions*, 1801, p. 111.

his *Anleitung zur Algebra.*[1] During the eighteenth century the Naperian definition, based on the two progressions, continued to be the prevailing definition. As late as 1808, C. F. Kaussler expressed decided preference for it, the new definition offering to beginners "gaps and obscurities."[2]

We have seen that Euler extended the exponential concept even to the use of imaginary exponents. We have seen that the expression $\sqrt{-1}^{\sqrt{-1}}$ is associated with his name; he recognized that it was infinitely many-valued. When the exponent is real and fractional, the multiple values of $a^{m/n}$ were recognized by Euler and others in research articles. As a rule, this many-valuedness was not discussed at all in text-books of the eighteenth century, such as the algebras of Saunderson, Blassière and Lacroix, or the mathematical texts of La Caille, J. F. Häseler, Abbé Sauri, Karsten, Reyneau, Bézout, Gherli, Rivard and Bertrand, or the mathematical dictionaries of E. Stone and Ch. Wolf.

The relation $a = e^{\log a}$ was easily deduced from the new definition of logarithms and was used by some writers of this century, such as G. Fontana in 1782.

[Further instalments will follow, carrying this history through the nineteenth century. EDITORS.]

SOME INVERSE PROBLEMS IN THE CALCULUS OF VARIATIONS.

By E. J. MILES, Yale University.

In the so-called inverse problem[1] of the calculus of variations a doubly infinite family of curves

$$y = f(x, \alpha, \beta) \tag{1}$$

is given and it is required to find a function $F(x, y, y')$ such that the given system of curves forms the system of extremals for the integral

$$I = \int_{x_0}^{x_1} F(x, y, y')dx. \tag{2}$$

Darboux[2] has shown that the problem always has an infinite number of solutions which can be reduced to quadratures. For if in the Euler differential equation

$$F_y - F_{y'x} - y'F_{y'y} - y''F_{y'y'} = 0$$

the expression $G(x, y, y')$, where $y'' = G(x, y, y')$ is the differential equation

[1] L. Euler, *Anleitung zur Algebra*, St. Petersburg, 1770, 1. Theil, Cap. 21.

[2] C. F. Kaussler, *Lehre von den Logarithmen*, Tübingen, 1808, preface. Quoted by Tropfke Gesch. d. Elem.-Math., 2. Band, Leipzig, 1903, p. 142.

[1] For a treatment of this problem see for instance Bolza: *Vorlesungen über Variationsrechnung*, § 6c.

[2] Darboux: *Theorie des Surfaces*, vol. III, Nos. 604, 605.

from which the curves (1) are obtained, is substituted for y'' then the resulting equation

$$F_y - F_{y'x} - y'F_{y'y} - G(x, y, y')F_{y'y'} = 0 \tag{3}$$

is satisfied identically in x, y and y'. Differentiating (3) as to y' and placing $M = F_{y'y'}$ a linear partial differential equation of first order is obtained, viz:

$$\frac{\partial M}{\partial x} + y'\frac{\partial M}{\partial y} + G\frac{\partial M}{\partial y'} + G_{y'}M = 0. \tag{4}$$

From this M may be obtained and then F by two quadratures.

Few examples illustrating this theory have been given. Hamel[1] has studied the case when the given curves are straight lines and Stromquist[2] the case when they are circles with center on the x-axis. The present note is intended to add further illustrative examples by considering first the case when the given curves are catenaries, secondly the case when the given curves are parabolas whose axes coincide with the x-axis, and finally the case when they are the ellipses or hyperbolas $x^2/\alpha + y^2/\beta = 1$.

§1. When the Given Curves are Catenaries.

It will be assumed that the catenaries have the x-axis for directrix. They therefore have the equation

$$y = \alpha \cosh \frac{x - \beta}{\alpha}.$$

The differential equation[3] from which this equation is obtained is

$$y'' = \frac{1 + y'^2}{y}.$$

Substituting this value of y'' the Euler equation then gives the following identity in x, y and y'

$$F_y - F_{y'x} - y'F_{y'y} - \frac{1 + y'^2}{y}F_{y'y'} = 0. \tag{5}$$

Hence M in this case must be obtained from the partial differential equation

$$\frac{\partial M}{\partial x} + y'\frac{\partial M}{\partial y} + \frac{1 + y'^2}{y}\frac{\partial M}{\partial y'} + \frac{2y'}{y}M = 0. \tag{6}$$

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[1] Hamel: *Über die Geometrien in denen die Geraden die Kurzesten sind;* dissertation, Göttingen, 1901; *Mathematische Annalen*, vol. 57 (1903), pp. 231–264.

[2] Stromquist: "*On geometries in which circles are the shortest lines,*" *Transactions of the American Mathematical Society*, vol. 7 (1906), pp. 175–183.

[3] This is readily verified when it is remembered that the projection of the ordinate of any point of the catenary on the normal to the catenary at this point is constant. Stated analytically this says that $y/\sqrt{1 + y'^2} = \alpha$, whence the above result.

To integrate this place $M = e^z$ and equation (6) becomes

$$\frac{\partial z}{\partial x} + y' \frac{\partial z}{\partial y} + \frac{1 + y'^2}{y} \frac{\partial z}{\partial y'} + \frac{2y'}{y} = 0. \tag{7}$$

Then if Φ_1 denotes an arbitrary function of the arguments indicated, the most general solution of equation (7) is found to be

$$\Phi_1 \left\{ \frac{y}{\sqrt{1 + y'^2}}, \quad x - \frac{y}{\sqrt{1 + y'^2}} \log (\sqrt{1 + y'^2} \pm y'), \quad z + \log (1 + y'^2) \right\} = 0$$

or

$$z + \log (1 + y'^2) = \Phi_2 \left\{ \frac{y}{\sqrt{1 + y'^2}}, \quad x - \frac{y}{\sqrt{1 + y'^2}} \log (\sqrt{1 + y'^2} \pm y') \right\}.$$

Hence M assumes the form

$$M = \frac{1}{1 + y'^2} \Phi \left\{ \frac{y}{\sqrt{1 + y'^2}}, \quad x - \frac{y}{\sqrt{1 + y'^2}} \log (\sqrt{1 + y'^2} \pm y') \right\},$$

where again the function Φ is an arbitrary function of its arguments.

From this it follows that a first integration gives

$$\begin{aligned} \frac{\partial F}{\partial y'} &= \int \frac{1}{1 + y'^2} \Phi \left\{ \frac{y}{\sqrt{1 + y'^2}}, \quad x - \frac{y}{\sqrt{1 + y'^2}} \log (\sqrt{1 + y'^2} \pm y') \right\} dy' + \lambda(x, y) \\ &= \int_0^{y'} \frac{1}{1 + t^2} \Phi \left\{ \frac{y}{\sqrt{1 + t^2}}, \quad x - \frac{y}{\sqrt{1 + t^2}} \log (\sqrt{1 + t^2} \pm t) \right\} dt + \lambda(x, y). \end{aligned}$$

Finally on integrating a second time F is found to be of the form

$$\begin{aligned} F = \int_0^{y'} (y' - t) \frac{1}{1 + t^2} \Phi \left\{ \frac{y}{\sqrt{1 + t^2}}, \quad x - \frac{y}{\sqrt{1 + t^2}} \log (\sqrt{1 + t^2} \pm t) \right\} dt \\ + \lambda(x, y) y' + \mu(x, y), \end{aligned} \tag{8}$$

in which λ and μ are arbitrary functions of x and y.

Now every F which satisfies equation (5) necessarily satisfies equation (6) but not conversely. It is therefore necessary to find what conditions the functions λ and μ must satisfy in order that the function F given by (8) may satisfy the equation (5). When this value of F is substituted in equation (5) it must result in a function independent of y'. Consequently it may be evaluated by giving y' any definite value, say $y' = 0$. This gives us the relation

$$-\frac{1}{y} \Phi(y, x) + \mu_y - \lambda_x = 0. \tag{9}$$

If then $\bar{\lambda}$ and $\bar{\mu}$ are particular values of λ and μ which satisfy condition (9), the most general values of λ and μ are

$$\lambda = \bar{\lambda} + \frac{\partial V}{\partial y}, \quad \mu = \bar{\mu} + \frac{\partial V}{\partial x},$$

where V is an arbitrary function of x and y. If one of the functions $\bar{\lambda}$ and $\bar{\mu}$, say $\bar{\mu}$, is chosen as zero, equation (9) gives

$$\frac{\partial \bar{\lambda}}{\partial x} = -\frac{1}{y}\Phi(y, x),$$

and therefore

$$\bar{\lambda} = -\frac{1}{y}\int_{x_0}^{x_1} \Phi(y, x)dx.$$

Hence the function F has as its final form the following:

$$F = \int_0^{y'} (y' - t)\frac{1}{1+t^2}\Phi\left\{\frac{y}{\sqrt{1+t^2}}, \quad x - \frac{y}{\sqrt{1+t^2}}\log\left(\sqrt{1+t^2} \pm t\right)\right\}dt$$
$$- \frac{y'}{y}\int_{x_0}^{x}\Phi(y, x)dx + y'\frac{\partial V}{\partial y} + \frac{\partial V}{\partial x}.$$

This is the most general form which it can assume and have as extremals the given set of catenaries.

Suppose now that a further restriction is made, namely, that the extremals are to be perpendicular to their transversals. In order to assertain what the restriction is in this case it is only necessary to recall the general condition of orthogonality of extremals and transversals, viz:

$$F(x, y, y') = \psi(x, y)\sqrt{1 + y'^2}.$$

In case of the present problem this leads to the condition

$$\Phi\left\{\frac{y}{\sqrt{1+y'^2}}, \quad x - \frac{y}{\sqrt{1+y'^2}}\log\left(\sqrt{1+y'^2} \pm y'\right)\right\} = \frac{\psi(x, y)}{(1+y'^2)^{\frac{1}{2}}}.$$

But in order that there may be some relation between the expression $\psi(x, y)/\sqrt{1 + y'^2}$ and the arguments of Φ it is necessary that their Jacobian vanish identically in x, y and y', i. e.,

$$\begin{vmatrix} \dfrac{\psi_x}{\sqrt{1+y'^2}} & \dfrac{\psi_y}{\sqrt{1+y'^2}} & -\dfrac{y'\psi}{(\sqrt{1+y'^2})^3} \\[2ex] 0 & \dfrac{1}{\sqrt{1+y'^2}} & -\dfrac{yy'}{(\sqrt{1+y'^2})^3} \\[2ex] 1 & -\dfrac{1}{\sqrt{1+y'^2}}\log\left(\sqrt{1+y'^2} \pm y'\right) & \begin{matrix} \dfrac{yy'}{(\sqrt{1+y'^2})^3}\log\left(\sqrt{1+y'^2} \pm y'\right) \\ \mp \dfrac{y}{1+y'^2} \end{matrix} \end{vmatrix} \equiv 0.$$

Expanding this determinant and grouping like powers of y' it is found that the following equation results:

$$\pm y\psi_x + y'[y\psi_y - \psi] \equiv 0.$$

Hence

$$y\psi_x = 0, \qquad y\psi_y - \psi = 0$$

and therefore

$$\psi = ky.$$

Hence if the extremals are to be catenaries which are orthogonal to their transversals the function Φ must assume the form

$$\frac{ky}{\sqrt{1 + y'^2}}.$$

§ 2. The Case of Parabolas.[1]

Let the equation of the parabolas be taken in the form

$$y^2 = 2\alpha x + \beta.$$

Then

$$y'' = -\frac{y'^2}{y}$$

and equation (3) becomes in this case

$$F_y - F_{y'x} - y'F_{y'y} + \frac{y'^2}{y} F_{y'y'} = 0. \tag{10}$$

Hence M is determined by the following equation:

$$\frac{\partial M}{\partial x} + y'\frac{\partial M}{\partial y} - \frac{y'^2}{y}\frac{\partial M}{\partial y'} - \frac{2y'}{y} M = 0. \tag{11}$$

Integrating by the method of the preceding section it is easily verified that the most general solution of equation (11) is

$$M = y^2\Phi(yy',\ y^2 - 2xyy'),$$

where again Φ is an arbitrary function of its arguments.

Hence

$$\frac{\partial F}{\partial y'} = \int_0^{y'} y^2\Phi(yt,\ y^2 - 2xyt)dt + \lambda(x, y)$$

and therefore

[1] Only points in the upper half plane are considered, for the tangent at the vertex of the parabola $y^2 = 2\alpha x + \beta$ is infinite and this case cannot be treated by the usual methods of the calculus of variations when the curve is taken in the form $y = y(x)$. If then $x_0 \neq x_1$, and $y_0 \neq y_1$, there passes one and but one parabola of the above type through any two points in the upper half plane.

$$F = \int_0^{y'} (y' - t)y^2\Phi(yt, y^2 - 2xyt)dt + y'\lambda(x, y) + \mu(x, y),$$

where λ and μ are functions of x and y.

When this value of F is substituted in equation (10) giving an equation independent of y' it is found upon evaluating by assigning the value zero to y' that λ and μ must satisfy the relation

$$\frac{\partial \lambda}{\partial x} = \frac{\partial \mu}{\partial y}.$$

Hence the most general expressions for λ and μ are

$$\lambda = \frac{\partial V}{\partial y}, \quad \mu = \frac{\partial V}{\partial x},$$

V being an arbitrary function.

The most general solution of the problem is therefore

$$F = \int_0^{y'} (y' - t)y^2\Phi(yt', y^2 - 2xyt)dt + y'\frac{\partial V(x, y)}{\partial y} + \frac{\partial V(x, y)}{\partial x}.$$

§ 3. When the Given Curves are Ellipses or Hyperbolas.

In this case the equations of the curves will be taken in the form[1]

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1.$$

The differential equation belonging to these curves is found to be

$$y'' = \frac{yy' - xy'^2}{xy}.$$

Hence the partial differential equation from which M is obtained is

$$\frac{\partial M}{\partial x} + y'\frac{\partial M}{\partial y} + \frac{yy' - xy'^2}{xy}\frac{\partial M}{\partial y'} + \frac{y - 2xy'}{xy}M = 0. \tag{12}$$

By the process previously described the following expression is found for M

$$M = \frac{y}{x(y - xy')}\Phi\left(\frac{-y'}{x(y - xy')}, \frac{1}{y(y - xy')}\right).$$

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[1] Here again only points in the upper half plane are considered. Then if the point (x_1, y_1) is not on either of the lines

$$x + \frac{x_0}{y_0}y = 0, \quad x - \frac{x_0}{y_0}y = 0,$$

it follows that there is a unique determination of the constants $1/\alpha$ and $1/\beta$. Hence, barring these exceptional cases, it follows that through any two points of the upper half plane there passes one ellipse or one hyperbola defined by the equation $x^2/\alpha + y^2/\beta = 1$.

Hence by quadrature

$$F = \int_0^{y'} (y' - t) \frac{y}{x(y - xt)} \Phi \left(\frac{-t}{x(y - xt)}, \frac{1}{y(y - xt)} \right) dt + y'\lambda(x, y) + \mu(x, y).$$

It is easily proved that the functions λ and μ must again satisfy the condition

$$\frac{\partial \lambda}{\partial x} = \frac{\partial \mu}{\partial y}.$$

Therefore the most general solution for F is seen to be

$$F(x, y, y') = \int_0^{y'} (y' - t) \frac{y}{x(y - xt)} \Phi \left(\frac{-t}{x(y - xt)}, \frac{1}{y(y - xt)} \right) dt + y' \frac{\partial V}{\partial y} + \frac{\partial V}{\partial x}.$$

Furthermore it can be proved that in neither of the last two cases can the function Φ be so chosen that the transversals are perpendicular to the extremals.

THE PROBABILITY INTEGRAL DEDUCED BY MEANS OF DEVELOPMENTS IN FINITE FORM.

By EDWARD L. DODD, University of Texas.

In modern mathematics, expansions in *finite* form are constantly coming into greater favor in preference to infinite series from which the terms after the first two or three have been dropped.

The object of this paper is to express in modern form the deduction of the probability integral as given by J. Bertrand,[1] whose method follows Laplace's,[2]—though his integral is not quite the same.

Let p be the probability that an event will happen in a single trial, where $0 < p < 1$; and let $q = 1 - p$. Then the probability, P_r, that the event will happen exactly r times on s trials is

$$P_r = \frac{s!}{r!\,(s - r)!} p^r q^{s-r}. \tag{1}$$

Here r is a positive integer not greater than s, or r is zero, accepting unity as the value of $0!$

Now let

$$l = sp - r. \tag{2}$$

Then $r = sp - l$, and

$$P_r = \frac{s!}{(sp - l)!(sq + l)!} p^{sp-l} q^{sq+l}. \tag{3}$$

[1] *Calcul des Probabilités* (1889), p. 76.

[2] *Œuvres Complètes*, VII, Theorie Analytique des Probabilités, p. 284.

Stirling's[1] Formula, in *finite* form is

$$n! = n^n e^{-n}\sqrt{2\pi n}\, e^{\theta/12n}, \quad 0 < \theta < 1. \tag{4}$$

Using (4), and taking r as a positive integer *less* than s, (3) becomes

$$P_r = \frac{1}{\sqrt{2\pi spq}} \frac{1}{\left(1 - \frac{l}{sp}\right)^{sp-l+\frac{1}{2}}} \frac{1}{\left(1 + \frac{l}{sq}\right)^{sq+l+\frac{1}{2}}} e^{\frac{\theta}{12s} - \frac{\theta'}{12(sp-l)} - \frac{\theta''}{12(sq+l)}}. \tag{5}$$

Now let any positive number c be chosen; and then let s be taken large enough so that

$$sp - c\sqrt{s} > 0, \quad \text{and} \quad sp + c\sqrt{s} < s. \tag{6}$$

And let r be such that

$$sp - c\sqrt{s} \leqq r \leqq sp + c\sqrt{s}. \tag{7}$$

Then

$$|\, l \,| \leqq c\sqrt{s}. \tag{8}$$

For brevity, set

$$u = \left(1 - \frac{l}{sp}\right)^{sp-l+\frac{1}{2}}, \quad v = \left(1 + \frac{l}{sq}\right)^{sq+l+\frac{1}{2}}.$$

Now, if $|\, t \,| < \frac{1}{2}$,

$$\log_e (1 + t) = t - \frac{t^2}{2} + \theta_1 t^3, \quad 0 < \theta_1 < 1. \tag{9}$$

But, if s is sufficiently great, l/sp and l/sq will be numerically less than $\frac{1}{2}$, because of (8). Then, by the use of (9), it may be shown that

$$\log_e uv = \frac{l^2}{2spq} + R, \qquad |\, R \,| < \frac{k}{\sqrt{s}},$$

in which k is some constant; that is, k is independent of s. Indeed, to get an upper bound for R, let all its terms be made positive, let the θ_1 in (9) be taken as unity, and let l be replaced by $c\sqrt{s}$, as in (8). In the expression thus obtained, the numerators are functions of c, and each denominator contains the factor $\sqrt{s}$.

Then,

$$uv = e^{\frac{l^2}{2spq}} \cdot e^{\frac{\theta_2 k}{\sqrt{s}}}, \quad -1 < \theta_2 < 1.$$

Now if $|\, t \,| < \frac{1}{2}$,

$$e^t = 1 + 2\eta t, \quad 0 < \eta < 1. \tag{10}$$

Hence, when s is large enough,

$$e^{-\frac{\theta_2 k}{\sqrt{s}}} = 1 + \frac{2\eta' k}{\sqrt{s}}, \quad -1 < \eta' < 1.$$

[1] See Cesaro, *Corso di analisi algebrica*, Turin, 1884, pages 270 and 480; or Broggi, *Traité des Assurances sur la Vie*, Trad. S. Lattès, 1907, p. 54.

The relation (10) can also be applied to the last factor in (5), noticing that in $sp - l$, the l becomes insignificant in comparison with sp when s is large, because of (8). Hence

$$P_r = \frac{1}{\sqrt{2\pi spq}} e^{-\frac{l^2}{2spq}} + \epsilon_1, \quad | \epsilon_1 | < \frac{K_1}{s}, \tag{11}$$

in which K_1 is some constant; that is, K_1 is independent of s.

For brevity, set

$$h = \frac{1}{\sqrt{2spq}}. \tag{12}$$

Let r_1 and r_2 be any two values of r which satisfy (7), with $r_1 > r_2$; and let l_1 and l_2 correspond by (2). Then,

$$\sum_{r_2}^{r_1} P_r = \frac{h}{\sqrt{\pi}} \sum_{l_1}^{l_2} e^{-h^2 l^2} + \epsilon_2, \quad | \epsilon_2 | < \frac{K_2}{\sqrt{s}}, \tag{13}$$

in which K_2 is some constant.

By dropping ϵ_1 in (11), we get an approximation, P_r', for P_r; viz.,

$$P_r' = \frac{h}{\sqrt{\pi}} e^{-h^2 l^2}. \tag{14}$$

Since r is an integer, any two consecutive values of l differ from each other by unity, because of (2). Use the values of l as abscissas and the corresponding values of P_r' as ordinates. With each of these ordinates as a central line, construct a rectangle with its base upon the x-axis, the base being unity in length. The *probability curve*, whose equation is

$$y = f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}, \tag{15}$$

passes through the middle points of the upper bases of these rectangles. The probability curve has but two points of inflection, and near the origin is concave towards the x-axis. For this portion of the curve, the following figure may be drawn. See page 126.

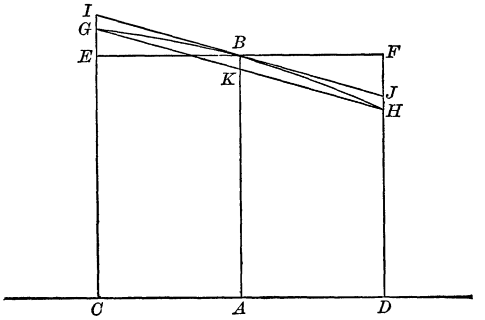
In this figure, A is the point on the X axis, whose abscissa is l; AB the ordinate whose length is P_r'; $CDFE$ the rectangle—bisected by AB—with its area numerically equal to P_r'; GBH an arc of the probability curve (15); GH a chord; IBJ the tangent to GBH at B; and K the intersection of AB and GH.

It is now to be proved that if l satisfies (8),

$$P_r' = \frac{h}{\sqrt{\pi}} \int_{l-1/2}^{l+1/2} e^{-h^2 x^2} dx + \epsilon_3, \quad | \epsilon_3 | < \frac{K_3}{s\sqrt{s}}, \tag{16}$$

in which K_3 is some constant.

Proof.—In the figure, the trapezoid $CDJI$ has an area numerically equal to P_r'; and this differs from the area beneath the arc GBH by less than the area of the trapezoid $GHJI$. This last mentioned area is $\overline{KB} \times \overline{CD} = \overline{KB}$; since $\overline{CD}$ is of unit length. Hence $\overline{KB}$ is numerically greater than ϵ_3 in (16).



But

$$\overline{KB} = f(l) - \tfrac{1}{2}[f(l - \tfrac{1}{2}) + f(l + \tfrac{1}{2})]$$
$$= \frac{he^{-h^2l^2}}{2\sqrt{\pi}}\left[2 - e^{-\frac{h^2}{4}}(e^{h^2l} + e^{-h^2l})\right]. \quad (17)$$

Moreover, if $|t| < \frac{1}{2}$,

$$e^t = 1 + t + \zeta t^2, \qquad 0 < \zeta < 1. \quad (18)$$

And, from (8) and (12), it is evident that $|h^2l|$ becomes less than $\frac{1}{2}$ when s is large enough. Using (18) upon the parenthesis in (17) and also upon $e^{-h^2/4}$, and noticing that $e^{-h^2l^2} \leqq 1$, we prove (16). Beyond the points of inflection the curve (15) is convex towards the X axis; but here, also, a similar treatment applies.

Then, from (11), (14), and (16), we can conclude that

$$\sum_{r_2}^{r_1} P_r = \frac{h}{\sqrt{\pi}}\int_{l_1-\frac{1}{2}}^{l_2+\frac{1}{2}} e^{-h^2x^2}dx + \epsilon_4, \quad |\epsilon_4| < \frac{K_4}{\sqrt{s}}, \quad (19)$$

in which K_4 is some constant. The discussion has been carried out under the restriction imposed by (8); viz.,

$$-c\sqrt{s} \leqq l_1 < l_2 \leq c\sqrt{s};$$

but the effect of this restriction may be removed by choosing c large enough.

Indeed, let $\alpha = l_2 + \frac{1}{2}$, suppose this positive and nearly equal to $c\sqrt{s}$; and, for convenience, suppose that we can take $l_1 - \frac{1}{2} = -\alpha$. Then the integral in (19) becomes $\Theta(h\alpha)$. Now, by (12), $\alpha < c\sqrt{s}$, provided

$$h\alpha < \frac{c}{\sqrt{2pq}}.$$

But, by taking c large enough, we may take as large values of $h\alpha$ as we wish, making $\Theta(h\alpha)$ differ from unity by as little as we please. For example, if we take $c = 5\sqrt{2pq}$, we may take s and α so that $4.9 < h\alpha < 5$; and $\Theta(h\alpha)$ will differ then from unity by less than 10^{-11}. Hence, by properly choosing c and then s, and taking r_2 and r_1 nearly equal to $sp - c\sqrt{s}$ and $sp + c\sqrt{s}$, respectively, $\sum_{r_2}^{r_1} P_r$ in (19) may be made as near unity as we please. But, by (1), $\sum_0^s P_r = (p + q)^s = 1$. Hence, the rectangles which the restriction (8) excludes are negligible when s is large enough.

By reference to (11), it is evident that no essential change in (19) is made if $l_1 - \frac{1}{2}$ and $l_2 + \frac{1}{2}$ are replaced by l_1 and l_2. respectively

THEOREM: *Let p be the probability that an event will occur in a single trial, where $0 < p < 1$; let s be the number of trials to be made; and let $h = [2sp(1-p)]^{-\frac{1}{2}}$. Then the probability that the number of occurrences of the event will be some number in the set, $sp - l_2$ to $sp - l_1$ inclusive, where $0 \leqq sp - l_2 \leqq sp - l_1 \leqq s$, is*

$$\frac{h}{\sqrt{\pi}} \int_{l_1}^{l_2} e^{-h^2x^2} dx + \delta,$$

where $\lim\limits_{s=\infty} \delta = 0$ *uniformly.*

While this commonly accepted theorem is the most important result of the foregoing set of inequalities, the relations, (11), (13), (16), and (19), are not without some individual importance.

WESTERN MEETINGS OF MATHEMATICIANS

The American Mathematical Society has been for nearly a quarter of a century the official organization through which scientific activity in mathematics has found expression in this country. In 1894 The New York Mathematical Society, which had then been in existence for six years, was expanded into the American Mathematical Society, and the *Bulletin*, which had been published for three years by the former, was continued by the latter as "A Historical and Critical Review of Mathematical Science."

The regular meetings of the Society were held in New York, except in the case of the summer meetings, which, from the first in 1894 to the nineteenth in 1912, were held respectively in Brooklyn, N. Y., Springfield, Mass., Buffalo, N. Y., Toronto, Can., Boston, Mass., Columbus, O., New York, N. Y., Ithaca, N. Y., Evanston, Ill., Boston, Mass., St. Louis, Mo., Williamstown, Mass., New Haven, Conn., Ithaca, N. Y., Urbana, Ill., Princeton, N. J., New York, N. Y., Poughkeepsie, N. Y., and Philadelphia, Pa. It thus appears that of the nineteen summer meetings, four have been held in the middle west, namely, at Columbus, in 1899, at Evanston in 1902, at St. Louis in 1904, and at Urbana in 1908.

Another notable gathering of mathematicians took place at Chicago in 1893, namely, the International Mathematical Congress held in connection with the World's Columbian Exposition. The papers read at this meeting, including nine by Americans and thirty by representatives of other countries, were published for the Society by the Macmillan Company in 1896. There was also held at Evanston, Ill., from August 28 to September 9, 1893, a Colloquium in connection with the World's Fair, consisting of a series of lectures delivered by Professor Felix Klein, of Göttingen, Germany. These lectures have been published by the Society. Other colloquia have been held from time to time in connection with the summer meetings, at which series of lectures have been delivered by members of the Society, but none of these have as yet been held in the west. The next one, however, is to be at Madison, Wis., in September, 1913, in connection with the twentieth summer meeting of the Society.

In 1896 a conference was called in Chicago by an informal committee of twenty-eight members of the Society to consider the desirability of organizing a Section of the Society in the middle west, in order that the stimulus and inspiration arising from attendance upon frequent meetings might be brought within the reach of those in this vicinity, it being quite impossible for most members in the west to attend the meetings in New York. This conference was held at the University of Chicago, in December, 1896, when a temporary organization was effected and fourteen papers were read. A second conference was called April 24, 1897, at the University of Chicago, when a communication from the Council of the Society was received authorizing the formation of a Section. The formal organization was thus completed and Professor E. H. Moore was made chairman, and Professor T. F. Holgate secretary, of the Chicago Section of the American Mathematical Society. Ten papers were read at this meeting Since that time thirty-one meetings of the Chicago Section have been held twenty-four at the University of Chicago, five at Northwestern University, one at Armour Institute, and one at the University of Minnesota in affiliation with the American Association for the Advancement of Science. The total attendance upon these meetings has aggregated 930 members and 288 visitors, and the total number of papers read is 569.

Two other Sections of the Society have been formed in the west, namely, the San Francisco Section, which meets alternately at the University of California and at Stanford University, and the Southwestern Section, which meets at various points including St. Louis, Mo., Columbia, Mo., Lawrence, Kans., and Lincoln, Neb. The latter was organized in November, 1907, and has held in all six meetings at which 79 papers have been read. The former was organized in May, 1902, and has held twenty-two meetings at which 182 papers have been read. The scientific activity shown by the records of these sectional meetings has amply justified the formal action of the Society as stated in By-Law IV, which says:

> "Whenever it shall appear to the Council that a sufficient number of members of the Society are desirous of conducting in any locality periodic meetings for the reading and discussion of mathematical papers, the Council may authorize the formation of a Section to be composed at each sectional meeting of such members of the Society as may be present; and the Council shall have the right to withdraw such authorization."

On two notable occasions regular meetings of the Society as a whole have been held in the middle west, and the meetings of the Chicago Section, which would have occurred at the same times and places, were merged in those of the Society. One of these meetings was held in Chicago, on the invitation of the Chicago Section, in April, 1911. It was the first regular meeting of the Society, aside from the summer meetings, to be held away from New York City since its founding there in 1888. It was arranged that this reunion of the eastern and western members should be especially marked by the delivery of President Bôcher's retiring address, which had been postponed from the annual meeting in the preceding December. The attendance upon this meeting numbered 115

persons, including 88 members of the Society who had come from as far east as Massachusetts, as far west as Wyoming, as far north as Minnesota and as far south as Missouri. There were 53 papers read at this meeting aside from Professor Bôcher's address on "Charles Sturm's published and unpublished work on differential and algebraic equations."

The other occasion when a regular meeting of the Society was held in the west was the Cleveland meeting in December, 1912, in affiliation with the American Association for the Advancement of Science.[1] There were 54 papers read in the mathematical sessions on this occasion and 62 members were in attendance.

Perhaps the most important occasion in connection with these gatherings was the series of meetings of mathematicians and engineers held at the University of Chicago in December, 1907, under the auspices of the Chicago Section of the American Mathematical Society, and conjointly with Sections A and D (mathematics and engineering) of the American Association for the Advancement of Science. The sole topic of these meetings was the discussion of the teaching of mathematics to students of engineering, by mathematicians, professors of engineering, and those engaged in the practice of engineering. The attendance was large and representative, including 100 mathematicians and fifty engineers from 23 state universities and technical schools and 12 other universities and colleges. The presentation of papers occupied three half days and resulted in the appointment of a committee of twenty members with Professor E. V. Huntington, of Harvard University, as chairman, to take into consideration the whole question of the mathematical curriculum in technical schools and in technical departments of colleges and universities, and to make their report to the Society for the Promotion of Engineering Education. That report has since been completed in the form of a syllabus of mathematical courses for students of engineering, which can be purchased through the aforesaid Society at 43 E. Nineteenth St., New York City.

The last meeting of the Chicago Section was held at the University of Chicago, March 21–22, 1913. There were 39 papers read during the four half-day sessions and the total attendance numbered 81 persons of whom 51 were members of the Society. The former officers of the Section were reëlected, namely, Professor D. R. Curtiss, Chairman, Professor H. E. Slaught, Secretary, and Professor A. L. Underhill, third member of the Program Committee. The sentiment of the Section was formally expressed in a resolution favoring the custom of having a special address by the chairman upon his retirement from office. The next occasion of the kind will occur at the December meeting, 1913. The last address of this kind was two years ago when Professor L. E. Dickson spoke on the "History of the representations of numbers as the sum of squares." It was the feeling of the members that we need more papers giving a general survey of some broad field and that this custom will serve, to some extent, to meet this desire.

A characteristic feature of the Chicago Section meetings is the full and free discussion of all papers presented, and especially the regular topic on the pro-

[1] Reported by Professor G. A. Miller in the February ,1913, issue of the MONTHLY.

gram called "Informal Notes and Queries," under which heading opportunity is given for discussion of any questions of mathematical interest that may not arise in connection with the formal papers. Another feature which is emphasized in these meetings is the dinner on the evening between the two days assigned to the program. These dinners are held at the Quadrangle Club of the University of Chicago where ample facilities are provided for social intercourse throughout the evening.

In closing, it should be said that, while the various sections were organized to accommodate members in their attendance upon meetings, there are no designated members of any given sections, but the membership at any meeting of a section consists of those members of the Society who happen to be in attendance. Thus a nation-wide brotherhood of mathematicians is included in the one American Mathematical Society.

H. E. SLAUGHT.

BOOK REVIEWS.

W. H. BUSSEY, CHAIRMAN OF THE COMMITTEE.

Solid Geometry developed by the Syllabus Method. By EUGENE RANDOLPH SMITH. American Book Company, New York, 1913. xii + 211 pages. $0.75.

This is a book for the teacher of solid geometry who wishes to try something out of the ordinary. The propositions are not proved in full and formal fashion, but in the opinion of the author the book contains as much suggestion and guidance as the student needs. The aim of the book is to encourage original thinking and to reduce mere memory work to a minimum. There are three preliminary chapters, one on the relation between solid geometry and plane geometry, one on methods of attack, and one on the representation of solid figures by drawing. At the end of the book there are 23 pages of college examination questions selected from the questions set by the College Entrance Examination Board and by various colleges.

W. H. B.

Memoranda Mathematica. A synopsis of facts, formulae, and methods in elementary mathematics. By W. P. WORKMAN. Clarendon Press, Oxford, 1912. iv + 272 + 28 pages. $1.75.

Although this book was written especially for the use of the English student, any teacher of college mathematics will find it a valuable addition to his library. The author says: "Most students and teachers of mathematics have at times wished for a book to which they might speedily refer for a particular formula or method. Such a want the author attempts to supply in the present volume, which collects in convenient compass the facts, formulae, and results of elementary mathematics and is intended, not to replace existing text-books, but to be a companion to them all, useful for revision and handy for reference."

The words "elementary mathematics" must be interpreted to mean all the subjects usually given in the first three years of an American college course, except integral calculus, which is entirely omitted. Many of the subjects are outlined much more fully than is usual in American texts. Not only are the formulas stated, but in many cases the methods of obtaining them are outlined with sufficient fullness to enable any teacher to complete them. As an illustration of the book's completeness, ten methods of solving numerical equations are given, some of which are entirely unknown to many teachers of college mathematics in this country.

Algebra, including the theory of equations and determinants, and plane and spherical trigonometry occupy the first 100 pages. Another 100 pages are devoted to the subject of geometry, from which the usual high school course in plane and solid geometry is omitted. Besides the ordinary course in analytic geometry, it includes the outline of what is usually given under the name of modern geometry: contact of conics, geometric and analytic treatment of polar reciprocation, harmonic ranges and pencils, inversion, etc.

The portion of the book devoted to the geometric treatment of conics is so complete that it might almost be used as a text-book in that subject, and the application of the differential calculus to the study of curves contains much that is not usually found in American books. The book closes with short outlines of statics, dynamics, and hydrostatics. Although the treatment throughout is distinctly of the English type, any American student specializing in mathematics will find it very useful.

C. H. Ashton.

ISIS, Revue consacrée a l'histoire de la science, publiée par George Sarton, D.Sc., Wondelgem-lez-Gand, Belgium.

This proposed journal, to be devoted to the history of science, has secured the support of a notable list of scholars representing almost every phase of academic activity. The "Comité de patronage" includes, as representatives of science, Svante Arrhenius, Jacques Loeb, Wilhelm Ostwald, Sir William Ramsay, Jean Mascart and others; among the representatives of mathematics and the history of mathematics are Moritz Cantor, Siegmund Günther, Sir Thomas L. Heath, H. G. Zeuthen and David Eugene Smith; among orientalists and classical scholars are Franz Cumont, Ludwig Stein and J. L. Heiberg; and among historians, Henri Berr and Karl Lamprecht.

The aim of the journal is to study the genesis and development of scientific ideas as an integral part of the history of civilization. Further than this the journal will assist in assembling the material necessary for such study and in perfecting the methods and instruments of historical research in science. It should appeal not only to historians of science but to all students of history, science, philosophy and sociology; in a word, to all who desire to understand any phase of the evolution of the human intellect.

Contributions are to be published in French, English, German or Italian.

This corresponds to the practise of the *Bibliotheca Mathematica* which is devoted to the history of the mathematical sciences. The editor of *Isis* is George Sarton, Wondelgem-lez-Gand, Belgium. Subscriptions, 30 francs per year for four numbers of about 160 pages each, should be sent to the printer, Soc. an. M. Weissenbruch, imprimeur du Roi, 49, rue du Poinçon, Brussels. The attention of our readers is called to the desirability of supporting such worthy projects for the advancement of science as *Isis* and the *Bibliotheca Mathematica* which latter is published by Teubner, Leipzig, at 20 marks per year.

L. C. KARPINSKI.

An Introduction to Mathematical Physics. By R. A. HOUSTON. Longmans, Green, and Co., London, 1912. x+200 pages. $2.00.

This book is intended as a first survey of the field of mathematical physics, for students already familiar with elementary physics and dynamics. As a single volume of this grade on the whole subject it is something of a novelty, at least in English. To compare with it the reviewer calls to mind only the somewhat similar works by Christiansen and Weinstein. Its scope is, however, more restricted than the title might seem to imply.

There is no consideration given to the molecular theory of matter, the modern electron theory, or the principle of relativity. Historical perspective is absent and the physical setting of the mathematical notions is treated in the briefest way. The book may perhaps be best described as a somewhat detailed syllabus of the classic mathematical physics of the eighteenth and nineteenth centuries, so far as that relates to the descriptive theory of physical media, expressed in differential equations and the related integral formulas.

There are six chapters, entitled: attraction, hydrodynamics, Fourier's series and conduction of heat, wave motion, electromagnetic theory, and thermodynamics. In view of the moderate space employed, the selection of material seems judicious and representative. The mathematical treatment is rather better than the explanations of experimental results. A few of the equations are written in vector form, as an alternative short-hand, but there is confusion in the notation between a vector and its absolute value, and no real use of vector analysis as an instrument of thought. Each chapter contains several lists of examples as exercises, on the whole free from the common fault of possessing merely academic interest; but no hint of a bibliography appears.

A. C. LUNN.

PROBLEMS AND QUESTIONS.

B. F. Finkel, Chairman of the Committee.

PROBLEMS FOR SOLUTION.

ALGEBRA.

387. Proposed by H. C. FEEMSTER, York, Neb.

Sum the following series:

$$(1)\ \sum_{n=1}^{n=\infty}\frac{1}{8^{n+1}n!};\qquad (2)\ \sum_{n=1}^{n=\infty}\frac{1.3.5.\cdots 2n+1}{2^{2n+1}(2n-1)!};\qquad (3)\ \sum_{n=1}^{n=\infty}\frac{1}{2^{2n}n!}$$

GEOMETRY.

415. Proposed by WILLIAM D. CAIRNS, Oberlin College.

For three points on a straight line, P_1, P_2, P_3, we have

$$P_1P_2 = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2};\qquad P_2P_3 = \sqrt{(x_3-x_2)^2+(y_3-y_2)^2+(z_3-z_2)^2};$$

and

$$P_1P_3 = \sqrt{(x_3-x_1)^2+(y_3-y_1)^2+(z_3-z_1)^2},$$

while

$$P_1P_2 + P_2P_3 = P_1P_3.$$

Prove directly that the sum of the first two radical expressions equals the third.

416. Proposed by W. H. BUSSEY, University of Minnesota.

Let AB be a diameter of any circle, and let C be the third vertex of the equilateral triangle of which AB is one side. Divide AB into six equal parts and let P denote the second point of division from A (*i. e.*, $AP = \frac{1}{3}AB$). Draw the straight line CP and produce it to meet the circle in R Prove that AR is one-sixth of the circumference.*

CALCULUS.

337. Proposed by R. P. BAKER, University of Iowa.

Show that for a, b relatively prime integers ,

$$\int_0^1(\cos 2\pi ax+\cos 2\pi bx)dx = \frac{2}{\pi ab}\left[\frac{a+b}{\sin\dfrac{\pi}{a+b}} - \frac{a-b}{\sin\dfrac{\pi}{a-b}}\right],$$

or

$$\frac{1}{\pi ab}\left[(a+b)\cot\frac{\pi}{2(a+b)} - (a-b)\cot\frac{\pi}{2(a-b)}\right]$$

according as a and b are both odd or one of them is even.

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* *Note of explanation:* This problem was called to my attention by a member of the class in design in our art department. A generalization of the construction can be used to divide a circle approximately into any number of equal parts. If the diameter AB is divided into n equal parts, and if P be the second point of division from A, the line CP produced will meet the circle in R so that the arc AR is an arc of $360°/n$. The construction is theoretically exact when $n = 6$ and in the trivial cases when $n = 2$ or 4. It is used, I am told, when $n = 5$ and 7 and the construction is very approximately correct. The problem for $n = 6$ is not difficult if one uses analytic geometry or trigonometry. Perhaps some reader of the Monthly can work it by means of elementary geometry. I asked the student who called the problem to my attention why she did not use the ordinary geometric constructions for dividing a circle into 5 or 6 equal parts. She said she wanted *one* construction that would be good in all cases.

W. H. B.

338. Proposed by RICHARD LOCHNER, Philadelphia, Pa.

An elliptical field has a major axis of 100 feet and a minor axis of 10 feet. A cow is tethered at the end of the major axis and another at the end of the minor axis. If each cow can graze over half the field, how long is the rope of each? What is the area of the portion over which the cows can graze in common?

MECHANICS.

274. Proposed by G. B. M. ZERR.

A sphere moves on the concave side of a rough cylindrical surface of which the transverse section perpendicular to the generating lines is a hypocycloid. If $s = l \sin n\theta$ be the intrinsic equation of the hypocycloid, then $l = (a - b)4b/a$, $n = a/(a - 2b)$, where $a =$ radius of fixed circle, $b =$ radius of rolling circle.

275. W. J. GREENSTREET, Editor Mathematical Gazette, England.

If a particle be attracted towards the angular points of a regular hexagon by forces equal to r^{-h}, at distance r, find the condition for stability of equilibrium.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

190. Proposed by H. C. FEEMSTER, York, Neb.

Show that, if n is a prime number and r is an integer less than n, then

$$\underline{|r-1}\ \underline{|n-r} + (-1)^{r-1} = M \cdot n,$$

where M is an integer.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

376. Proposed by W. W. BEMAN, Ann Arbor, Mich.

If

$$\left(1 + \frac{1}{m}\right)^m \Big/ e = 1 - a_1 \frac{1}{m} + a_2 \frac{1}{m^2} - a_3 \frac{1}{m^3} + \cdots,$$

prove that

$$na_n = \sum_{k=1}^{k=n} \frac{k}{k+1} a_{n-k}$$

and compute $a_1, a_2, a_3, \cdots, a_8$.

SOLUTION BY THE PROPOSER.

For convenience, put $1/m = x$, and $y = (1 + x)^{1/x}/e$. Then

$$ey = (1 + x)^{1/x},$$

$$\log y = -\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots$$

and

$$\frac{y'}{y} = -\frac{1}{2} + \frac{2x}{3} - \frac{3x^2}{4} - \cdots.$$

Assume

$$y = 1 - a_1 x + a_2 x^2 - a_3 x^3 + \cdots.$$

Then

$$\begin{aligned} y' &= -a_1 + 2a_2 x - 3a_3 x^2 + 4a_4 x^3 - \cdots \\ &= \left(1 - a_1 x + a_2 x^2 - a_3 x^3 + \cdots\right)\left(-\frac{1}{2} + \frac{2x}{3} - \frac{3x^2}{4} + \cdots\right). \end{aligned}$$

Equating coefficients,

$$a_1 = \tfrac{1}{2}, \qquad 2a_2 = \tfrac{1}{2}a_1 + \tfrac{2}{3},$$
$$3a_3 = \tfrac{1}{2}a_2 + \tfrac{2}{3}a_1 + \tfrac{3}{4},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$na_n = \tfrac{1}{2}a_{n-1} + \tfrac{2}{3}a_{n-2} + \tfrac{3}{4}a_{n-3} + \cdots + \frac{n}{n+1}$$
$$= \sum_{k=1}^{k=n} \frac{k}{k+1} a_{n-k}.$$

The first six values of a_n are

$$a_1 = \tfrac{1}{2}, \quad a_2 = \frac{11}{24}, \quad a_3 = \frac{7}{16}, \quad a_4 = \frac{2447}{5760}, \quad a_5 = \frac{459}{2304}, \quad a_6 = \frac{238043}{580608}.$$

See solution of 377 in the December issue.

Solutions of 377 and 378 were received, too late for credit in the December issue, from C. N. Schmall and H. E. Trefethen; also of 378 from A. L. McCarty.

GEOMETRY.

409. Proposed by S. LEFSCHETZ, University of Nebraska.

The sides of a triangle being in arithmetic progression, a and a' being, respectively, the smallest and largest sides, r, R the radii of the inscribed and circumscribed circles, to prove that $6Rr = aa'$.

SOLUTION BY W. C. EELLS, Tacoma, Wash.

Since the sides are in arithmetic progression, represent them by

$$a, \quad \frac{a+a'}{2}, \quad a'.$$

Then from trigonometry, we have

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \quad \text{and} \quad R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

where s is the half-sum of the three sides a, b, c.

Substituting $s = \dfrac{3(a+a')}{4}$ and reducing, we have

$$r = \frac{1}{4}\sqrt{\frac{(3a'-a)(3a-a')}{3}}, \quad \text{and} \quad R = \frac{2aa'}{\sqrt{3(3a'-a)(3a-a')}}$$

whence $rR = \dfrac{aa'}{6}$ and $6rR = aa'$.

Also solved by J. Scheffer, C. N. Schmall, Levi S. Shively, A. M. Harding, and A. L. McCarty.

410. Proposed by A. H. HOLMES, Brunswick, Me.

Given a focus and two tangents to an ellipse, prove that the locus of the foot of the normal corresponding to either tangent is a straight line.

No solution has been received.

411. Proposed by C. N. SCHMALL, New York City.

$ABCD$ is a rectangle of known sides. BC being produced indefinitely, it is required to draw a straight line from A cutting CD and BC in X and Y, respectively, so that the intercept XY may be equal to a given straight line. (Unsolved in *Educational Times.*)

REMARK BY L. S. SHIVELY, Mt. Morris, Ill.

This problem cannot be solved with straight edge and compasses only, for an arbitrary length of the given straight line. To prove this, it will be sufficient to show that for a particular length the solution, with these restrictions, is impossible.

Let the length of the given line be twice the diagonal of the rectangle. Suppose the problem to have been solved. Let O be the midpoint of XY. Draw AC and CO. Then triangles COY and ACO are isosceles and we have

$$\angle \text{CAO} = \angle \text{AOC} = 2 \angle \text{OYC}$$

and

$$\angle \text{ACB} = \angle \text{CAO} + \angle \text{OYC} = 3 \angle \text{OYC}.$$

Since the given rectangle is arbitrary we have here a solution for the trisection of an arbitrary acute angle, which is known to be impossible with straight edge and compasses. Hence the proposed construction is also impossible.

Solutions were received, too late for credit in the December number, of 403, from W. H. Johnson; of 404, from S. Lefschetz; of 406, 407, and 408 from M. A. Muzzy and H. E. Trefethen.

CALCULUS.

330. Proposed by C. N. SCHMALL, New York City.

X is a homogeneous function, in the nth degree, of x, y, z; Y is any function of u, v, w. If $ux = vy = wz = 1 + X$...(1), prove that (by Eurler's Theorem),

$$x\frac{\partial Y}{\partial x} + y\frac{\partial Y}{\partial y} + z\frac{\partial Y}{\partial z} + \frac{n}{x}\frac{\partial Y}{\partial u} + \frac{n}{y}\frac{\partial Y}{\partial v} + \frac{n}{z}\frac{\partial Y}{\partial w} = (n-1)\left(u\frac{\partial Y}{\partial u} + v\frac{\partial Y}{\partial v} + w\frac{\partial Y}{\partial w}\right).$$

SOLUTION BY THE PROPOSER.

From (1) we have

$$\log x + \log u = \log y + \log v = \log z + \log w = \log (1 + X)$$

Differentiating, we have

$$\frac{1}{x} + \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{1+X}\frac{\partial X}{\partial x},$$

or

and similarly,

$$\left.\begin{aligned}\frac{x}{u}\frac{\partial u}{\partial x}&=\frac{x}{1+X}\frac{\partial X}{\partial x}-1\\ \frac{y}{u}\frac{\partial u}{\partial y}&=\frac{y}{1+X}\frac{\partial X}{\partial y}\\ \frac{z}{u}\frac{\partial u}{\partial y}&=\frac{z}{1+X}\frac{\partial X}{\partial z}\end{aligned}\right\}\tag{2}$$

We get similar expressions involving the partial derivatives of v and w, with respect to x, y and z.
Now

$$\begin{aligned}x\frac{\partial Y}{\partial x}+y\frac{\partial Y}{\partial y}+z\frac{\partial Y}{\partial z}+\frac{n}{x}\frac{\partial Y}{\partial u}+\frac{n}{y}\frac{\partial Y}{\partial v}+\frac{n}{z}\frac{\partial Y}{\partial w}\equiv\frac{\partial Y}{\partial u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)&\\ +\frac{n}{x}\frac{\partial Y}{\partial u}+\frac{n}{y}\frac{\partial Y}{\partial v}+\frac{n}{z}\frac{\partial Y}{\partial w}.&\end{aligned}\tag{3}$$

From (1),

$$x=\frac{1+X}{u},\quad y=\frac{1+X}{v},\quad z=\frac{1+X}{w}.\tag{4}$$

Hence, the right member of (3) can be written, in the form,

$$\frac{\partial Y}{\partial u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)+\frac{nu}{1+X}\frac{\partial Y}{\partial u}+\frac{nv}{1+X}\frac{\partial Y}{\partial v}+\frac{nw}{1+X}\frac{\partial Y}{\partial w}.\tag{5}$$

But by adding the three equations in (2) we have

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=\frac{x}{1+X}\frac{\partial X}{\partial x}+\frac{y}{1+X}\frac{\partial X}{\partial y}+\frac{z}{1+X}\frac{\partial X}{\partial z}-1.\tag{6}$$

Now by the aid of (6) the expression (5) can be written,

$$u\frac{\partial Y}{\partial u}\left(\frac{x}{1+X}\frac{\partial X}{\partial x}+\frac{y}{1+X}\frac{\partial X}{\partial y}+\frac{z}{1+X}\frac{\partial X}{\partial z}-1\right)+\frac{n}{1+X}\left(u\frac{\partial Y}{\partial u}+v\frac{\partial Y}{\partial v}+w\frac{\partial Y}{\partial w}\right)\tag{7}$$

$$=u\frac{\partial Y}{\partial u}\left(\frac{nX}{1+X}-1\right)+\frac{n}{1+X}\left(u\frac{\partial Y}{\partial u}+v\frac{\partial Y}{\partial v}+w\frac{\partial Y}{\partial w}\right)\tag{8}$$

(by applying Euler's Theorem to the first parenthesis).

$$=u\frac{\partial Y}{\partial u}\left(\frac{nX-X-1}{1+X}\right)+\frac{n}{1+X}\left(u\frac{\partial Y}{\partial u}+v\frac{\partial Y}{\partial v}+w\frac{\partial Y}{\partial w}\right),$$

$$=\frac{1}{1+X}\left\{u\frac{\partial Y}{\partial u}(nX-X-1)+n\left(u\frac{\partial Y}{\partial u}+v\frac{\partial Y}{\partial v}+w\frac{\partial Y}{\partial w}\right)\right\}.$$

Remark.—The right member of the result I first gave is apparently incorrect.

Also solved by Elmer Schuyler.

331. Proposed by T. H. GRONWALL, Chicago, Illinois.

To show that:

(1) $\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right) = \frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y + \frac{n+1}{2}\pi\right) dy,$

(2) $\frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right) = \frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y + \frac{n}{2}\pi\right) dy.$

SOLUTION BY ELIJAH SWIFT, Princeton, N. J.

Assume that the formula is true for $n - 1$.

Then

$$\frac{d^{n-1}}{dx^{n-1}}\left(\frac{\sin x}{x}\right) = \frac{1}{x^n}\int_0^x y^{n-1} \sin\left(y + \frac{n}{2}\pi\right) dy.$$

Differentiating both sides and using the formula we have to prove

$$(A) \quad \frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y + \frac{n+1}{2}\pi\right) dy = -\frac{n}{x^{n+1}}\int_0^x y^{n-1}\sin\left(y+\frac{n}{2}\pi\right) dy + \frac{1}{x^n}\cdot x^{n-1}\sin\left(x + \frac{n}{2}\pi\right).$$

or

$$(B) \quad \int_0^x y^n \sin\left(y + \frac{n+1}{2}\pi\right) dy = -n\int_0^x y^{n-1}\sin\left(y+\frac{n\pi}{2}\right) dy + x^n \sin\left(x + \frac{n}{2}\pi\right).$$

Differentiating B as to x,

$$(C) \quad x^n \sin\left(x + \frac{n+1}{2}\pi\right) = -nx^{n-1}\sin\left(x+\frac{n\pi}{2}\right) + nx^{n-1}\sin\left(x+\frac{n\pi}{2}\right) + x^n\cos\left(x+\frac{n\pi}{2}\right).$$

which is obviously an identity.

Hence (B) is an identity, except for an additive constant. This constant must be zero, as we see if we let x approach zero. (A) follows from (B). Then since the given formula holds for $n = 0$, it holds for all values of n.

Formula (2) may be derived in the same manner.

Also solved by A. M. Harding, W. C. Eells and the Proposer.

NEWS AND NOTES.

FLORIAN CAJORI, CHAIRMAN OF THE COMMITTEE.

The March number of *Popular Science Monthly* contains an article on "Henri Poincaré as an Investigator" by PROFESSOR JAMES B. SHAW.

PROFESSOR G. A. MILLER, of the University of Illinois, will teach in the summer session of the University of California.

A campaign is in progress to raise a fund of about $100,000 for the purpose of endowing a second professorship of natural philosophy at the University of Edinburgh, as a memorial to PROFESSOR PETER GUTHRIE TAIT who died in 1901, after a long career as professor at Edinburgh. To mathematicians Tait is best known for his book on quaternions and his topological researches on knots.

PROFESSOR CONSTANTIN CARATHÉODORY has been appointed professor of mathematics at the University of Göttingen as successor to Professor Felix Klein. Carathéodory made his doctorate at Göttingen in 1904, was privatdozent at Göttingen and Bonn, then professor at Hanover and Breslau.

PROFESSOR VIVANTI, University of Pavia, Italy, has recently published a volume of about 500 pages, entitled "Esercizi di analisi infinitesimale." The book is composed of 575 problems and their solutions. The problems are chosen especially with a view to applications in mechanics and in physics, and more than two-thirds of them are said to be new. A large number of these exercises relate to finding the derivative or the integral. The work closes with solutions of differential equations and a few problems in the calculus of variations. The arrangement of the material is similar to that of the "Lezioni di analisi infinitesimale," which was published by the same author in 1911.

The following suggestion from a new subscriber, whose attention was arrested by the Foreword in the January issue, may contain some pedagogical significance: ". . . In short why not get contributions from those who do not like mathematics, telling just what it was about their college course in mathematics which was distasteful to them. This might include not only college students at the present time in advanced classes, but also men and women who have excelled in other lines but did not prefer mathematics. Such contributions, of course, would be of little interest to mathematicians, but they would no doubt bring to light some matters very useful to those concerned with the teaching of mathematics. I am one who believes the popular distaste for mathematics is brought about largely by poor teaching methods, and that in the large majority of cases the students' dislike may be traced to instructors' methods rather than to anything in the mathematics itself." A really scientific investigation along this line might lead to valuable results.

See also the leading article in this issue, entitled "Some things we wish to know."

Attention is called to the advertisements of publishers and others in the various issues of the MONTHLY. The EDITORS desire to express their appreciation of the interest shown by the representatives of these firms in thus contributing toward the support of this journal. We trust that our readers will reciprocate this interest by referring to these announcements whenever possible.

SUMMER COURSES IN MATHEMATICS FOR 1913.

UNIVERSITY OF CHICAGO. June 16 to August 29. See the February issue.

UNIVERSITY OF ILLINOIS. June 16 to August 8.—By Dr. Crathorne, College

algebra (6 hours), Calculus of variations (6 hours); By Dr. Börger, Trigonometry (5 hours), Theory of equations (6 hours); By Mr. Rutledge; Analytic geometry (10 hours); By Mr. Rowland, First course in calculus (10 hours); By Mr. Carscallen, Second course in calculus (6 hours). These courses are equivalent in credit and in number of class room hours to the corresponding courses given in the regular university year.

University of Kansas. June 12 to July 23.—By Assistant Professor Duval, Solid geometry (2 hours), Higher algebra II (3 hours, graduate credit); By Associate Professor Van der Vries, Plane trigonometry (2 hours), Projective geometry (2 hours, graduate credit); By Assistant Professor Jordan, Analytic geometry I (2 hours), Calculus I (3 hours). It is planned to offer a series of six different graduate courses in three consecutive summer sessions, giving a total of 15 hours credit toward the A.M. degree.

University of Michigan. June 21 to August 22.—By Professor W. W. Beman, Differential equations, Geometry and algebra (Teachers' Course); By Professor J. L. Markley, Theory of functions, Differential calculus; By Professor W. B. Ford; Infinite series and products, Harmonic analysis; By Professor P. Field; Projective geometry, College algebra; By Professor T. R. Running, Differential and integral calculus; By Professor L. C. Karpinski, History of mathematics (arithmetic and algebra), Analytic geometry (double course); By Mr. Escott, Theory of finance, insurance and statistics, Trigonometry; By Mr. Garretson, Solid geometry, Solid analytic geometry; By Mr. Coe, Elementary algebra, Plane geometry; By Mr. Hopkins, Calculus.

University of Minnesota.—By Dean J. F. Downey, Higher algebra, part I, Higher algebra, part II; By Professor W. E. Brooke, Trigonometry and analytic geometry, Differential and integral calculus; By Professor W. H. Kirchner, Solid geometry, Descriptive geometry; By Dr. H. L. Slobin, Trigonometry, Analytic geometry.

University of Wisconsin. June 23 to August 1.—By Professor Van Vleck, College algebra, Axioms of geometry, Principal functions of mathematics and physics; By Assistant Professor Hart, Elementary solid geometry, Analytic geometry, The teaching of secondary mathematics; By Assistant Professor Dowling, Plane trigonometry and logarithms, Calculus, Introduction to higher plane curves; By Assistant Professor Dresden, Theory of complex numbers and algebraic equations, Differential equations, Calculus of variations; By Mr. Burgess, Elementary analysis (begins June 16), Elementary theory of transformations; By Mr. Taylor, Integral calculus (begins June 19).

The University of California.—By Assistant Professor J. H. McDonald, Solid geometry, Theory of functions of a complex variable; By Associate Professor D. N. Lehmer, Plane analytic geometry, Differential calculus; By Professor G. A. Miller, Plane trigonometry, Theory of algebraic equations; By Mr. B. A. Bernstein, Graphic algebra, Integral calculus; By Miss Thirmuthis Brookman, Mathematical teaching in the high school, Teachers' course in mathematics for secondary schools.

A FEW TYPICAL AUTHORITIES

Out of 1500 Specialist Contributors to

THE NEW ENCYCLOPAEDIA BRITANNICA

(Published by the Press of the University of Cambridge, England) From the field of

Mathematics and Physical Science

(Including Seven Winners of the Nobel Prize)

Major P. A. MacMahon,
Joint General Secretary, British Association, writes *Algebraic Forms; Combinatorial Analysis* and the biography of *Arthur Cayley.*

Prof. A. N. Whitehead,
of Cambridge, author of "A Treatise on Universal Algebra," contributes *Geometry (non-Euclidean, Axioms); Mathematics.*

Prof. E. B. Elliott,
of Oxford, author of "Algebra of Quantities," etc., writes on *Curves; Geometry (Analytical).*

Prof. A. E. H. Love,
of Oxford, contributes *Elasticity; Function (Functions of Real Variables): Infinitesimal Calculus; Variations, Calculus of.*

Prof. William Burnside,
of the Royal Naval College, Greenwich, author of "The Theory of Groups of Finite Order," deals with *Theory of Groups.*

Prof. H. F. Baker,
of Cambridge, author of "Abel's Theorem and the Allied Theory," etc., writes on *Differential Equations; Functions of Complex Variables.*

Dr. E. W. Hobson,
of Cambridge, contributes *Fourier's Series; Spherical Harmonics; Trigonometry.*

Sir A. G. Greenhill,
author of "Differential and Integral Calculus with Applications"; "Hydrostatics"; etc., contributes *Ballistics; Gyroscope and Gyrostat; Hydromechanics.*

Prof. Horace Lamb,
of Manchester, author of "Hydrodynamics," etc., writes on *Dynamics; Harmonic Analysis; Mechanics, theoretical; Vector Analysis; Wave.*

Sir George H. Darwin,
(d. 1912), of Cambridge, author of "Tides and Kindred Phenomena in the Solar System," etc., has revised and amplified his epoch-making article in the 10th Edition, *Tide.*

Prof. H. L. Callendar,
writes the articles *Thermodynamics; Calorimetry; Calibration; Conduction of Heat; Fusion; Heat; Thermoelectricity; Thermometry; Vaporization.*

Lord Rayleigh,
Nobel prize winner for Physics, 1904, writes on *Argon; Sky; Capillary Action; Diffraction of Light; Interference of Light.*

Prof. H. A. Lorentz,
Nobel prize winner for Physics, 1902, noted for researches in spectroscopy and light, writes on *Light, Nature of.*

Sir J. J. Thomson,
Nobel prize winner for Physics, 1906, eminent for discoveries relating to the theory of electrons, writes on *Matter; Electric Waves; Röntgen Rays; Conduction (electric, through gases); Magneto-Optics; Vacuum Tube.*

Dr. Wilhelm Ostwald,
Nobel prize winner for Chemistry, 1909, the effects of whose teachings regarding "matter" and "energy" are recognized as of fundamental importance to science, writes the article *Element.*

Prof. Ernest Rutherford,
Nobel prize winner for Chemistry, 1908, writes on *Radio-Activity,* the subject of which he is a leading investigator.

Prof. J. H. van't Hoff,
(d. 1911), Nobel prize winner for Physics, 1901, celebrated for his researches in physical chemistry, wrote *Isomerism* and *Stereo-isomerism.*

Prof. J. D. van der Waals,
Nobel prize winner for Physics, 1910, writes *Condensation of Gases.*

Sir James Dewar,
who first obtained liquid and solid hydrogen, contributes *Liquid Gases.*

Sir Joseph Larmor,
Lucasian Professor of Mathematics, Cambridge, author of "Aether and Matter," writes on *Aether; Energetics; Energy; Radiation, Theory of; Radiometer; Units, Dimensions of.*

"The feeling which one has that in consulting these volumes he will obtain an absolutely authoritative opinion is alone worth the cost of the work."—Henry Crew (*Professor of Physics, Northwestern University*).

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THE AMERICAN MATHEMATICAL MONTHLY

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THE FOUNDATION PERIOD IN THE HISTORY OF GROUP THEORY.

By JOSEPHINE E. BURNS, University of Illinois.

The Earliest Group Notions.—Henri Poincaré has pointed out that the fundamental conception of a group is evident in Euclid's work; in fact that the foundation of Euclid's demonstrations is the group idea. Poincaré establishes this assertion by showing that such operations as successive superposition and rotation about a fixed axis presuppose the displacements of a group. However much the fundamental group notions were unconsciously used in the work of early mathematicians, it was not until the latter part of the eighteenth century that these notions began to take life and develop.

The Foundation Period.—The period of foundation of group theory as a distinct science extends from Lagrange (1770) to Cauchy (1844–1846), a period of seventy-five years. We find Lagrange considering the number of values a rational function can assume when the variables are permuted in every possible way. With this beginning the development may be traced down through the contributions of Vandermonde, Ruffini, Abbati, Abel, Galois, Bertrand and Hermite, to Cauchy's period of active production (1844–1846). At the beginning of this period group theory was a discovery useful in the theory of equations; at the end it existed as a distinct science, not yet, to be sure, entirely free but so nearly so that this may be called the close of the foundation period.

LAGRANGE, RUFFINI, GALOIS.

Lagrange.—The contributions of Lagrange are included in his memoir, *Reflexions sur la résolution algêbrique des equations*, published in the Mémoirès of the Academy of Science at Berlin in 1770–1771.[1] In this paper Lagrange first applies what he calls the "calcul des combinaisons" to the solution of algebraic equations. This is practically the theory of substitutions, and he uses it to show wherein the efforts of his predecessors, Cardan, Ferrari, Descartes, Tschirnhaus, Euler and Bezout fail in the case of equations of degree higher than the

[1] *Œuvres de Lagrange*, vol. 3, pp. 204–420.

fourth. He studies the number of values a rational function can assume when its variables are permuted in every possible way. The theorem that the order of the subgroup divides the order of the group is implied but not explicitly proved. The theorem that the order of a group of degree n divides $n!$ is however explicitly stated. Lagrange does not use at all the notation or the terminology of group theory but confines himself entirely to direct applications to the theory of equations. He mentions the symmetric group of degree 4, and the four-group and the cyclic group of order 4 as subgroups of this group. This embodies practically all of the work of Lagrange in the theory of substitution groups.

Ruffini.—The first man to follow Lagrange and to make any signal progress was Paolo Ruffini, an Italian, who published in 1799 in his *Teoria generale della equazioni* a number of important theorems in the theory of substitutions. Burkhardt says[1] that several fundamental concepts are implied, if not explicitly stated. Ruffini's "permutation" corresponds to the later accepted term of group and to Cauchy's "system of conjugate substitutions." These permutations he classifies as "simple" and "complex." The first are of two sorts, of one cycle or of more than one cycle. The second he divides into three classes which correspond to the modern notions of (1) intransitive, (2) transitive imprimitive, and (3) transitive primitive. He enunciates the following theorems:

1. If a substitution of n letters leaves the value of a rational function invariant, the result of applying the substitution any number of times is that the function is left invariant.
2. The order of a substitution is the least common multiple of the orders of the cycles.
3. There is not necessarily a subgroup corresponding to every arbitrary divisor of the order of the group.

This theorem he established by showing that there is no eight-valued, four-valued or three-valued function on five letters.

4. A group of degree five that contains no cycle of degree five cannot have its order divisible by five.

Abbati, Abel.—From Ruffini to Galois, only two or three contributions of merit were made. Abbati gave the first complete proof that the order of the subgroup divides the order of the group. This he did by putting the substitutions of the group in rectangular array. Abbati also proved that there is no three or four valued function on n letters when n is greater than five. Cauchy published in 1815 a relatively unimportant memoir. Abel, by using the theory of substitutions to prove that it is impossible to solve algebraically equations of degree higher than the fourth, called attention to this useful instrument.

Galois.—Up to the time of Cauchy, Galois had done the most for group theory. His work was written in 1831 and 1832, but not made public until 1846. To Galois is due the credit for the conception of the invariant subgroup, for the notion of the simple group and the extension of the idea of primitivity. Galois first used the term group in its present technical sense. Several important theorems are due to him. Among these are the following:

1. The lowest possible composite order of a simple group is 60.[2]
2. The substitutions common to two groups form a group.[3]

---

[1] *Zeitschrift für Math. u. Phys.* (1892), 37, p. 119.

[2] *Œuvres Mathématiques d' Everiste Galois*, p. 26.

[3] *Manuscrits de Éveriste Galois*, p. 39.

3. If the order of a group is divisible by p, a prime, there is at least one substitution of order p.
4. The substitutions of a group that omit a given letter form a group.

Cauchy, the Founder of Group Theory.

Terminology.—Cauchy's work on groups is found in his *Exercises d'analyse et de physique mathematique* published in 1844, and in the series of articles published in the Paris *Comptes Rendus* in 1845–1846.[1] Cauchy defines permutation and substitution in the same way, but uses the latter term almost entirely. He uses several devices to denote a substitution, the most common being (x, y, z, u, v, w) where each letter is replaced by the one which follows it and the last by the first. He defines this as a cyclic substitution. In the article written in 1815[2] he defines the degree of a substitution as the first power of the substitution that reduces to identity, but later he also defines order in this way and uses the word order rather than degree in his subsequent work. He uses both the term unity and identical substitution. A transposition is defined and the terms similar, regular, inverse and permutable substitutions are found with their present day significance. A group he calls a "system of conjugate substitutions," which may be transitive or intransitive. A transitive imprimitive group is "transitive complex." The order of a system of conjugate substitutions is the number of substitutions that it contains. The term "divisor indicatif" is also found for the order of a group. Cauchy frequently transfers his definition of order to the theory of equations and speaks of the number of equal values a rational function can assume when the variables are permuted in every possible way. The number of distinct values of such a function is the index. In the early article the order of operation is from left to right, but in all his later work he reverses the order and operates from right to left.

The Memoir of 1815.—In the paper published in 1815 the one theorem which is of special interest is that the number of distinct values of a non-symmetric function of degree n cannot be less than the largest prime that divides n, without becoming equal to 2. This theorem is proved. He states the special cases, that if the degree of a function is a prime number greater than 2, the number of distinct values cannot be less than the degree; and that if the degree is 6 the number of distinct values cannot be less than 6. He makes special reference to the functions belonging to (1) the intransitive group of degree 6 and order 36; (2) the transitive imprimitive group of degree 6 and order 72; (3) the intransitive group of degree 6 and order 48; (4) and the symmetric group of degree 5 considered as an intransitive group of degree 6. The functions are as follows:

(1) $a_1a_2a_3 + 2a_4a_5a_6$, (3) $a_1a_2a_3a_4 + a_5a_6$,

(2) $a_1a_2a_3 + a_4a_5a_6$, (4) $a_1a_2a_3a_4a_5 + a_6$.

The Memoir of 1844.—In the memoir published in 1844 in his *Exercices d'Analyse* there is much more of importance. The first theorem proved is that

[1] *Œuvres de A. Cauchy*, First series, vols. 9 and 10.

[2] *Journal de l'Ecole polytechnique*, 10 (1815), p. 1.

every substitution similar to a given substitution P is the product of three factors, the extremes of which are the inverse of each other and the middle term of which is P. Conversely, every product of three factors, the first and last of which are the inverse of each other, is similar to the middle term P. An important formula developed is for the number of substitutions similar to a given substitution. If ω is the number required, n the total number of letters and P the given substitution, composed of f cycles of order a, g cycles of order b, h cycles of order c, etc., and r the number of letters fixed in P, then

$$\omega = \frac{n!}{(f!)(g!)(h!) \cdots (r!)a^f b^g c^h \cdots}.$$

Among other theorems that he proves in this article are the following:

1. If P is a substitution of order i, h any number and θ the highest common factor of h and i, then P^h is of order i/θ.
2. If P is a substitution of order i, the substitutions among the powers of P that are of order i are the powers of P whose indices are prime to i. These substitutions are likewise similar to P, and hence the number of substitutions similar to P among the powers of P are in number equal to the number of numbers less than i and prime to it.
3. Let P be any substitution, regular or irregular; let i be its order and p any prime factor of i. Then a value for i can always be found such that P^i is a substitution of order p.
4. A substitution and its inverse are always similar.
5. The powers of a cycle constitute the totality of substitutions that transform the given cycle into itself.
6. The order of a system of conjugate substitutions is divisible by the order of each substitution.
7. Two permutable substitutions with no common power but identity, together generate a system of conjugate substitutions whose order is the product of the orders of the two substitutions.
8. If $1, P_1, P_2, \cdots P_{a-1}$ and $1, Q_1, Q_2, \cdots Q_{b-1}$ are two groups, one of order a and the other of order b, which are permutable and have no common terms but identity, then the group generated by these groups is of order ab.
9. The converse of (8).
10. If P and Q are two substitutions, one of order a and the other of order b, and if the two series $Q, PQ, P^2Q, \cdots P^{a-1}Q$ and $Q, QP, QP^2, \cdots QP^{a-1}$ are made up of the same terms in the same or different order, then the cyclic group generated by P is permutable with the cyclic group generated by Q.

A most important element in establishing Cauchy's claim as the founder of group theory is the proof which he gives of the fundamental theorem that if m is the order of a group and p any prime which divides m, there is at least one substitution of order p. Galois had stated the theorem but had not proved it. Its importance is due to the fact that it is the first step toward Sylow's theorem which appeared nearly thirty years later. In this memoir there are a number of specific groups mentioned although no enumeration of groups of special degrees is attempted. The substitutions of several groups are written out. Among them are (1) the octic, (2) the four-group, (3) the holomorph of the cyclic group of order five, (4) the intransitive group of degree 6 and order 9 and (5) the intransitive group of degree 6 and order 16. In addition to these groups the substitutions of which are given, several other groups are mentioned, among them (1) the holomorph of order 42, (2) the group of degree 7 and order 21, (3) the holomorph of the cyclic group of order 9.

There is also mentioned a group of degree 9 and order 27, generated by the two substitutions

$$P \equiv x_0x_3x_6 \cdot x_1x_4x_7 \cdot x_2x_5x_8 \quad \text{and} \quad Q \equiv x_1x_2x_4x_8x_5 \cdot x_3x_6.$$

This is obviously impossible but a little study of his method reveals that he should have used $Q \equiv x_1x_3x_4x_8x_7x_5 \cdot x_3x_6$, and that in order to get the order of the group he has incorrectly applied the theorems he was announcing and illustrating. The order of the group should be 18. The two theorems which he was illustrating may be stated as follows:

1. If P is a substitution on n letters, $P = x_0x_1x_2 \cdots x_{n-1}$, and if r is any number prime to n and if Q is a substitution derived from P by replacing each letter by another whose subscript is r times its own, then for any values of h and k

$$Q^kP^h = P^{r^kh}Q^k.$$

2. If in the above theorem we take v any divisor of n, distinct from unity, and let R be any substitution that replaces any letter x_l by the letter with the subscript $l + v$, then Q^kR^h generates a group of order vi, where i is the smallest value of k that satisfies the congruence $r^k \equiv i \pmod{n}$.

Since in the present instance r is 2, i must be 6. The order of the group is then $vi = 3 \cdot 6 = 18$, whereas Cauchy seems to take $i = 9$ and derives the order of the group $vi = 3 \cdot 9 = 27$.

The Memoirs of 1845–1846.—Some of the material in the memoirs[1] published in the *Comptes Rendus* had already appeared in the *Exercices*, but is treated more extensively here. There is much, however, in the later articles that is not touched in the *Exercices*. It is in the later memoirs that he first defines what he means by a system of conjugate substitutions. The definition is as follows:

I shall call derived substitutions all that can arise from the given substitutions by multiplying them one or more times by each other or by themselves; and the given substitutions together with all the derived substitutions form what I shall call a system of conjugate substitutions.

Then follow the general theorems:

1. If i is the order of a substitution P and $a, b, c, \cdots$ are the prime factors of i, then the substitution P and its powers form a group generated by the substitutions $P^{i/a}, P^{i/b}, P^{i/c} \cdots$.
2. If $P, Q, R, S, \cdots$ are the substitutions that leave a given function Ω invariant, they form a group whose order is the number of equal values of Ω when the variables are permuted in every possible way.
3. The order of a transitive group of degree n is n times the order of the subgroup that leaves one letter fixed.
4. If Ω is a transitive function on n letters, if m is the index of the corresponding transitive group under the symmetric group of degree n, then m will likewise be the index of the subgroup that leaves one letter fixed under the symmetric group of degree $n - 1$.

Intransitive Groups.—The general subject which Cauchy treats next is that of intransitive groups. He implicitly divides intransitive groups into two classes. The first class includes those in which the systems are independent, that is, in our terminology, those in which the systems of transitivity are united by direct product. The second class includes those in which the systems are dependent, that is, those in which the systems are united by some sort of iso-

[1] *Œuvres de A. Cauchy*, First series, volumes 9 and 10.

morphism. The concept of isomorphism is however only implied. He states first that an intransitive group is formed by the combination in some way of transitive constituents. He then develops some interesting formulas for the order of intransitive groups. If the systems are independent, then the order M is the direct product of $A, B, C, \cdots$ the orders of the transitive constituents and the index will be $n!/ABC \cdots$ where n is the total number of letters. This same formula holds also when the systems are not independent if $A, B, C \cdots$ are defined in a special way. These definitions are as follows:

Let A be the order of the first transitive constituent;

Let B be the number of substitutions involving letters in the second system without involving any of the first;

Let C be the number of substitutions involving letters in the third system, without involving any of the first or second, and so on.

With these definitions of $A, B, C, D \cdots$ the order of the intransitive group is $M = ABC \cdots$.

Imprimitive Groups.—Cauchy next turns to imprimitive groups. Here he confines himself almost entirely to the consideration of those imprimitive groups which are simply transitive and which have for the subgroup that leaves one letter fixed the direct product of the transitive constituents. In regard to this particular type of imprimitive groups, he gives several theorems which are of interest, even though he does not touch upon the broader and more general principles.

If in a simply transitive group the subgroup that leaves one letter fixed is an intransitive group formed by the direct product of its transitive constituents, the group is imprimitive. He considers the special cases $a > n/2$, $a < n/2$, $a = n/2$ in which a, the number of letters in the largest transitive constituent and n is the degree of the group.

The only theorems which he enunciates that apply to imprimitive groups in general are:

1. The number of letters in the systems of imprimitivity must be a divisor of the degree.
2. If A is the number of substitutions that permute the variables within the systems and K the number of ways the k systems can be permuted, then the order of the group is KA^k.

Symmetric Groups.—Cauchy then considers briefly symmetric groups. One theorem which he states with its corollaries is as follows:

If a transitive group of degree n has a symmetric subgroup of degree a, where $a > n/2$, then the group is symmetric on n letters.

For $n > 2$, a transitive group of degree n is symmetric if it contains a symmetric subgroup of degree $n - 1$.

If $n > 3$, a transitive group of degree n is symmetric if it contains a symmetric subgroup of degree $n - 2$. The special case $n = 4$ is excepted; for the octic group, belonging to the function $\Omega = xy + zu$ is symmetric on two letters, yet not symmetric on all four.

If $n > 4$, a transitive group of degree n is symmetric if it contains a symmetric subgroup of degree $n - 3$. The case $n = 6$ is excluded. A group requiring this exception is the imprimitive group of degree 6 and order 72 which contains the symmetric group of order 6.

If $n > 5$, a transitive group of degree n is symmetric if it contains a symmetric subgroup of degree $n - 4$. If $n = 6$ or $n = 8$, this does not hold. Cauchy gives as examples of the case $n = 6$ the imprimitive groups of degree 6 and orders 72 and 48.

Some Implied Theorems.—One of the most important and interesting parts of Cauchy's work on group theory is that in which he develops some complicated formulas from which we may easily deduce the theorem that the average number of letters in the substitutions of a transitive group is $n - 1$. Cauchy himself does not enunciate the theorem, although he brings the proof to the point where only the final statement is necessary to complete it. The formula for the case where the group is simply transitive is as follows:

$M = 2H_{n-2} + 3H_{n-3} + \cdots + (n - 2)H_2 + n$ where M is the order of the group, n its degree and H_{n-r} the number of substitution involving $n - r$ letters.

Further development of these formulas leads to the two conclusions:

1. If G is l-fold transitive of degree n, the number of substitutions involving $n - l + 1$ letters is equal to or greater than $n(n - 1) \cdots (n - l + 2)/(n - l)\ l!$;
2. If G is simply transitive, of degree n, the number of substitutions involving n letters is equal to or greater than $n - 1$.

We find here the assertion that if in an l-fold transitive group $l + 1$ letters are left fixed, all are left fixed. That this is false is demonstrated by considering the imprimitive group of degree 6 and order 72. It is simply transitive and hence $l = 1$; but $l + 1 = 2$ letters may be left fixed without all being so.

The theorem that the order of the holomorph of a cyclic group is the product of the orders of the cyclic group and its group of isomorphisms is found implicitly in Cauchy. He gives two theorems relative to this point.

1. Let $P = x_0x_1x_2 \cdots x_{n-1}$ be a substitution of order n. Let r be a primitive root of the modulus n, and I the smallest of the indices of unity belonging to the base r. Then let Q be the substitution that replaces x_l by x_{rl}. The order of Q will be I and the order of the group will be nI.

This I is nothing else than the order of the group of isomorphisms of the cyclic group. If n is a power of a prime p, then $I = n(1 - 1/p)$. When n is a prime then $I = n - 1$.

2. With the same hypothesis as in (1), P^a and Q^b, where a and b are divisors of n and I respectively, generate a group of order nI/ab.

Enumeration of Orders.—Cauchy was the first to attempt an enumeration of the possible orders of groups. This he did with a fair degree of accuracy up to and including the sixth degree. The enumeration including degree 5 is correct and complete, but several errors occur in the enumeration of those of degree 6. For instance, 150 is given as the index of a group of degree 6 under the symmetric group, although factorial six is not a multiple of 150.

Cauchy goes back to his original distinction between imprimitive groups with heads which are direct products and those with heads formed by isomorphisms. He gives as the possible orders of groups of degree 6 with heads the direct products of the transitive constituents, 72, 48, 24, 18, 16, and 8. The last two numbers are clearly impossible for there is no transitive group of degree 6 and order 8 or 16. The orders of the groups possible when the head is formed with isomorphisms are given as 6, 12, 4, while there is no imprimitive group of degree 6 and order 4. According to his classification there should also be included in this last enumeration 24 and 18. A complete omission occurs in the list of imprimitive groups since the groups of order 36 with two systems of imprimitivity are not mentioned. Otherwise the enumeration through degree 6 is correct and complete.

Errors of Cauchy.—The errors in Cauchy's work on group theory may be divided into two classes. The first and smaller class includes a number of serious errors in logic. The second class includes a large number of minor errors, some typographical, others arising through careless statement. All of the serious errors found have been noted in this paper with the exception of one to which attention has already been called.[1] He states an erroneous theorem on imprimitive groups in the following form.

If G_1 is a subgroup composed of all the substitutions that omit a given letter in a simply transitive group then G is imprimitive unless all the transitive constituents of G_1 are of the same degree.

Conclusion.—In conclusion we may say that the foundation period in the history of group theory includes the time from Lagrange to Cauchy inclusive; that at the beginning of this period group theory was a means to an end and not an end in itself. Lagrange and Ruffini thought of substitution groups only in so far as they led to practical results in the theory of equations. Galois, while broadening and deepening the application to the theory of equations may be considered as taking the initial step toward abstract group theory. In Cauchy while a group is still spoken of as the substitutions that leave a given function invariant, and the order of a group is still thought of as the number of equal values which the function can assume when the variables are permuted in every possible way, nevertheless quite as often a group is a system of conjugate substitutions and its relation to any function is entirely ignored.

In Cauchy's work it is to be noted that use is made of all the concepts originated by the earlier writers in substitution theory except those of Galois, a number of which would have proved powerful instruments in his hands, notably the idea of the invariant subgroup of which he makes no explicit use.

Because of the important theorems Cauchy proved, because of the break which he made in separating the theory of substitutions from the theory of equations and because of the importance that he attached to the theory itself, he deserves the credit as the founder of group theory.

HISTORY OF THE EXPONENETIAL AND LOGARITHMIC CONCEPTS.

By FLORIAN CAJORI, Colorado College.

V. Generalizations and Refinements Effected During the Nineteenth Century.

GRAPHIC REPRESENTATION.

The time of Wessel and Argand has been chosen as the beginning of a new epoch in our history, not because Wessel and Argand made any noteworthy advance in exponential and logarithmic theory, but because the graphic repre-

[1] See G. A. Miller in *Bibliotheca Mathematica* (1910), series 3, vol. 10, p. 321.

sentation of imaginaries for which the names of Wessel[1] and Argand[2] preëminently stand, led mathematicians gradually to recognize the *reality* of imaginaries and to feel the need of a more general and a more consequential development of the algebra of imaginaries. The graphic representation of imaginaries had engaged the attention of the Englishman, John Wallis, in the seventeenth century[3] and of the German W. J. G. Karsten in the eighteenth century. Wallis proposed different schemes, none of which satisfactorily visualized vector addition. As previously shown in this history, Karsten concentrated his effort upon the graphic representation of imaginary logarithms, rather than on imaginary numbers in general. In view of these facts the graphic representation due to Wessel and Argand, which was finally adopted by the rank and file of mathematicians through the commanding authority of C. F. Gauss,[4] was a real advance.

The Wessel-Argand diagram does not offer an obvious means of displaying the infinitely many-valued logarithms of a number. Half a century passed before a generalization of this diagram was effected for the display of all the logarithms. For the first half of the nineteenth century there existed no diagram which could take rank with that of the circular and hyperbolic arcs, invented by Karsten in the eighteenth century.

A necessary generalization of the Wessel-Argand diagram was effected by Bernhard Riemann through the surfaces now called "Riemann's surfaces." One purpose of these surfaces was the visualization of the different values of functions of complex variables. The logarithmic function was simply one of several elementary functions to which Riemann's idea applied. Karsten's scheme serves only for the logarithmic function and is therefore less general. Riemann's idea was outlined in his inaugural dissertation, in 1851, at Göttingen, in a paper bearing the title, *Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen complexen Grösse.*[5] The reader will picture to himself two planes, one representing the complex numbers z, the other the complex values w, where w is an n-valued function of z. To each point of the z-plane correspond n points of the w plane. It is a purpose of Riemann's scheme to create a geometric representation which avoids the multiplicity of values of a many-valued function and enables one to treat the function as if it were uniform. For this purpose he assumes the w plane to be made up of n sheets or leaves, which together represent the values of w. To each point in one sheet corresponds only one of w; to the n points lying one below the other correspond the n different values of w

[1] *Essai sur la représentation analytique de la direction*, par Caspar Wessel, Copenhague, 1897. Translation of Wessel's memoir in *K. Danske Videnskabernes Selskabs Skrifter, Femte Del. Kjobenhavn*, 1799.

[2] *Essai sur une manière de representer les quantités imaginaires dans les constructions géométriques*, Paris, 1806. An English translation, with historical notes, was brought out by A. S. Hardy, (Van Nostrand) New York, 1881.

[3] G. Eneström, *Bibliotheca mathematica*, 3d S., Vol. 7, 1906–'07, pp. 263–269; *Encyclopédie d. scien. mathématiques*, 1908, T. I., Vol. 1, 5, p. 339, note 51; F. Cajori, *Am. Math. Monthly*, Vol. XIX, 1912, pp. 167–171.

[4] See C. F. Gauss, *Werke*, Bd. II, Göttingen, 1876, p. 95.

[5] B. Riemann, *Math. Werke*, 2. Auflage, Leipzig, 1892, pp. 1–43. See particularly p. 41.

which are gotten from one value of z. If for a certain value of z several values of w become equal to each other, the sheets are supposed to be connected with each other, forming a *branch-point*. For the purpose of exhibiting the continuous passage of one value of w into another, so-called *branch-cuts* are made, which are lines drawn between any two branch-points or from a branch-point to infinity. These branch-cuts are supposed to break the conection, so that there is no passage across them from one side of a sheet to the other side of the same sheet, but they establish a connection between one side of a sheet and the opposite side of some other sheet. Which particular sheets should be connected, depends, of course, upon the properties of the function. These *branch-cuts* and the connections along the cross-cuts between different sheets establish one continuous surface. Riemann makes some brief statements concerning the logarithmic function, but makes no drawing to illustrate the surface to which it gives rise. Applying Riemann's general scheme to the function $w = \log z$, we see that to the z-plane there corresponds a w-surface of infinitely many sheets with all the sheets hanging together at 0 and ∞ and with a branch-cut drawn from 0 to ∞.

Special attention to the graphic representation of the infinitely many values of the general power u^v, where $u = a\ (\cos \alpha + i \sin \alpha)$ and $v = x + iy$, was given by John Warren[1] of Jesus College, Cambridge, and thirty-one years later by H. Durège[2] of the university of Prag. Durège refers to the work of Warren and reëstablishes the latter's result that the points in a plane representing the values of one and the same power all lie upon a logarithmic spiral and are so distributed that the radius vectors of two successive points include the constant angle $2\pi x$. Durège proceeds to a more elaborate study of the wonderful spirals and their vectors.

Somewhat similar studies which culminated in a new graphic representation of the infinitely many logarithms of a complex number—a representation that has found wide acceptance in texts on the theory of functions of complex variables and is often preferred to the visualization by Riemann's surfaces—was given in 1871 by G. Holzmüller of Magdeburg in a paper, *Ueber die logarithmische Abbildung und die aus ihr entspringenden orthogonalen Curvensysteme.* Holzmüller[3] takes $Z = \log z$ and shows that, to each infinitely long strip of area of the Z-plane, drawn parallel to the x-axis, and of width 2π, there corresponds the entire z-plane; that to each strip drawn parallel to the y-axis there corresponds in the Z-plane a circular ring of definite width, which must, however, be taken as wound about itself an infinite number of times. To each point of the Z-plane corresponds only one point of the z-plane; to each point of the z-plane corresponds an infinite number of points of the Z-plane, all of which lie on a line parallel to the y-axis and appear at intervals of 2π. Thus Holzmüller pictures the geometric signi-

[1] J. Warren published two articles in the *Philosophical Transactions* for the year 1829, London, pp. 241 and 339.

[2] H. Durège, in *Zeitsch. f. Math. u. Physik*, Leipzig, Vol. 5, 1860, p. 345.

[3] *Zeitschr. f. Math. u. Phys.*, Leipzig, Vol. 16, 1871, pp. 269–289.

cance of the many-valuedness of the logarithm and the periodicity of the exponential function. He acknowledges suggestions received from a lecture by Schwarz of Zürich. Holzmüller connects these results with the Riemann surfaces, and proceeds to elaborate some of the wonderful properties of the logarithmic spiral and to show the connection of this with the Ptolemaic stereographic projection and loxodromic lines.

The use of the logarithmic spiral in the graphic representation of complex numbers and their logarithms to any base, real or complex, upon the Wessel-Argand plane, is explained by Irving Stringham and M. W. Haskell in papers which we discuss more fully under the classification of logarithmic systems.

[The next instalment will treat the general power and logarithm.]

CERTAIN THEOREMS IN THE THEORY OF QUADRATIC RESIDUES.

By D. N. LEHMER, University of California.

The definition of a quadratic residue is usually given as follows: *If an integer* X *can be found to satisfy the congruence,*

$$X^2 \equiv D (\mathrm{mod}\ m)$$

where D *is relatively prime to* m, *then* D *is said to be a quadratic residue of* m. The restriction that D shall be relatively prime to m simplifies many results. There are theories, however, in which this restriction serves to cloud the results and in this paper we will not impose it. We proceed, first of all, to determine the number of residues for a given integer m, using the enlarged definition.

We take first of all the case where the modulus m is a power of an odd prime p. In the first place there will arise $\frac{1}{2}\varphi(p^\alpha)$ distinct residues from the squaring of the numbers k, which are less than p^α, and prime to p, $p^\alpha - k$ and k giving the same residues. Consider next the numbers kp, where k is prime to p. If two such numbers when squared give the same residue, we have:

$$k^2p^2 \equiv k_1^2p^2 (\mathrm{mod}\ p^\alpha),$$

whence,

$$k_1^2 \equiv k_1^2 (\mathrm{mod}\ p^{\alpha-2})$$

or

$$k \equiv \pm k_1 (\mathrm{mod}\ p^{\alpha-2}).$$

Give now k the $\varphi(p^{\alpha-2})$ values less than $p^{\alpha-2}$ and prime to p. The resulting squares furnish $\frac{1}{2}\varphi(p^{\alpha-2})$ distinct residues. In the same way the numbers kp^2 where k is prime to p furnish $\frac{1}{2}\varphi(p^{\alpha-4})$ distinct residues which also are different from those obtained from kp obtained above. Proceeding in this way we obtain for the total number of distinct residues:

$$\tfrac{1}{2}[\varphi(p^\alpha) + \varphi(p^{\alpha-2}) + \varphi(p^{\alpha-4}) + \varphi(p^{\alpha-6}) + \cdots] + 1$$

the unit being added for the residue zero.

For α even, the resulting formula is:

$$\frac{p^{\alpha+1}-p}{2(p+1)}+1 \quad (p \text{ an odd prime}).$$

For α odd:

$$\frac{p^{\alpha+1}-1}{2(p+1)}+1.$$

For m a power of 2, the result is somewhat different. There are $2^{\alpha-3}$ distinct residues resulting from the squares of the odd numbers. This is due to the congruence, $(2^{\alpha-2}-k)^2 \equiv (2^{\alpha-2}+k)^2 (\text{mod } 2^\alpha)$. There are $2^{\alpha-5}$ residues arising from the squares of the numbers $2k$ where k is odd. This comes from the congruence, $(2^{\alpha-3}-2k)^2 \equiv (2^{\alpha-3}+2k)^2 (\text{mod } 2^\alpha)$. Similarly there are $2^{\alpha-7}$ distinct residues arising from the squares of the numbers 2^2k where k is odd, and in general there are $2^{\alpha-(2\kappa+3)}$ residues arising from the squares of the odd multiples of 2^κ. Thus in case of α odd we have the series $2^{\alpha-3}+2^{\alpha-5}+2^{\alpha-7}+\cdots$, which terminates with 1, the last term resulting from odd multiples of 2 where $\lambda = (\alpha-3)/2$. There remain besides two residues; one resulting from 2^λ where $\lambda = (\alpha-1)/2$, and one from 2^λ where $\lambda = (\alpha+1)/2$. This last is seen to be zero, modulus 2, owing to the congruence,

$$(2^{\alpha+1/2}\cdot k)^2 \equiv 2^{\alpha+1}\cdot k^2 \equiv 0 (\text{mod } 2^\alpha).$$

For α odd therefore the formula is $(2^{\alpha-1}+5)/3$. For α even the result is $(2^{-3}+2^{\alpha-5}+2^{\alpha-7}+2)+2$; that is $(2^{\alpha-1}+4)/3$.

As illustrations of the results, the number of residues of the number $81 = 3^4$, is $(3^5-3)/2\cdot 4+1$ or 31. The actual residues are: 0, 1, 4, 7, 9, 10, 13, 16, 19, 22, 25, 28, 31, 34, 36, 37, 40, 43, 36, 39, 52, 55, 58, 61, 63, 64, 67, 70, 73, 76, 79.

For the number $27 = 3^3$ the number is $(3^4-1)/2\cdot 4+1$ or 11. The actual residues are: 0, 1, 4, 7, 9, 10, 13, 16, 21, 25.

For the number $32 = 2^5$ the formula gives $(2^4+5)/3 = 7$. The residues are: 0, 1, 4, 9, 16, 17, 25.

For the number $64 = 2^6$ we have $(2+4)/3 = 12$. The residues are: 0, 1, 4, 9, 16, 17, 25, 33, 36, 41, 49, 57.

For a modulus m which is the product of any number of powers of primes, the number of residues is obtained by taking the product of the number of residues for the separate powers of primes. Thus if $m = p_1^{\alpha_1}p_2^{\alpha_2}$, the solution of $X^2 \equiv D(\text{mod } m)$ is possible if and only if the separate congruences $X^2 \equiv D(\text{mod } p_1^{\alpha_1})$ and $X^2 \equiv D(\text{mod } p_2^{\alpha_2})$ are possible; and any solution of the first may be combined with any solution of the second to give a solution of the original congruence. Thus if $D \equiv \alpha(\text{mod } p_1^{\alpha_1})$ is such that the congruence $X^2 \equiv D(\text{mod } p_1^{\alpha_1})$ is solvable and $D \equiv \beta\ (\text{mod } p_2^{\alpha_2})$ such that the congruence $X^2 \equiv D\ (\text{mod } p_2^{\alpha_2})$ is solvable, then as the moduli are relatively prime to each other one and only one solution is possible of the two congruences, $D \equiv \alpha(\text{mod } p_1^{\alpha_1})$ and $D \equiv \beta(\text{mod } p_2^{\alpha_2})$ taken simultaneously and the resulting D makes $X^2 \equiv D\ (\text{mod } p_1^{\alpha_1}\, p_2^{\alpha_2})$ solvable. Thus the number of residues of the number 42 is equal to the product of the

number of residues of 2, 3 and 7. That is, $2 \cdot 2 \cdot 4$ or 16. The residues are in fact: 0, 1, 4, 7, 9, 15, 16, 18, 21, 22, 25, 28, 30, 36, 37, 39.

Consecutive Residues.—Certain interesting theorems[1] concerning the number of consecutive residues for a given prime number have been obtained for the current definition of a quadratic residue by M. Aladov[2] and also by M. von Sterneck.[3] The results of M. Aladov,—I have not seen the paper,—are as follows:

Let x = number of non-residues followed by a non-residue,
x' = number of non-residues followed by a residue,
y = number of residues followed by a non-residue,
y' = number of residues followed by a residue.

Then for p a prime of the form $4n + 1$ we have:

$$x = x' = y = \frac{p-1}{4}; \quad \text{and} \quad y' = \frac{p-5}{4}$$

and for p a prime of the form $4n + 3$ we have:

$$x = x' = y' = \frac{p-3}{4}, \quad \text{and} \quad y = \frac{p+1}{4}.$$

M. von Sterneck has extended these results to show that for every prime except 2, 3, 5, 7, 11, and 17, there is at least one group of four consecutive residues, or four consecutive non-residues.

M. Aladov's results are not difficult to obtain from a consideration of the congruence, $xy \equiv 1(\text{mod } p)$. This congruence groups the $p - 1$ numbers, $1, 2, 3, 4, \cdots p - 1$, in pairs. For two values of x, namely, ± 1, the value of y is equal to x. For other values of x the value of y is different. There are then in all $(p + 1)/2$ pairs. Let x', y', be such a pair. Then there is another pair $(p - x')(p - y')$, and for our purpose these two pairs are not essentially distinct, as they furnish identical pairs of consecutive residues as follows: If we put $a + b \equiv x'$ and $a - b \equiv y'$ we obtain $a^2 - b^2 \equiv 1(\text{mod } p)$, or a and b are consecutive residues. The pair $(p - x')(p - y')$ give the same pair of residues. The number of pairs of consecutive residues therefore would seem to be $(p + 1)/4$, and this is indeed the correct formula for p, a prime of the form $4n + 3$. For p a prime of the form $4n + 1$ the number of pairs will be odd and there will be one pair in which $x' = p - y'$, or in which the pair (x', y') is identical with the pair $(p - y', x')$. The number of distinct pairs of consecutive residues for such a prime is therefore $(p + 3)/4$. In these formulæ we have included the residue zero. Thus for the prime 11, we have the following pairs of values x, y and the corresponding values of a and b:

[1] These theorems have been proved by Jordan in his *Traite des Substitutions*, 1870, page 158 Professor Dickson has kindly called my attention to Jordan's proof, which is based on different principles from that given here.

[2] *Recueil Mathématique*, Société de Moscou, t. XVIII, 1895.

[3] *Ibid.*, t. XX, 1898.

x, y.	$p-x, p-y$.	Residues.
1, 1.	10, 10.	1, 0.
2, 6.	9, 5.	5, 4.
3, 4.	8, 7.	4, 3.

There are thus three consecutive pairs of residues for the prime 11. For the prime 17, we have the following pairs:

x, y.	$p-x, p-y$.	Residues.
1, 1.	16, 16.	1, 0.
2, 9.	15, 8.	9, 8.
3, 6.	14, 11.	16, 15.
4, 13.	13, 4.	17, 16.
5, 7.	12, 10.	2, 1.

Aladov's formula would give only three pairs because he does not admit the residue 0. This drops out the first and fourth pair.

The rest of M. Aladov's results follow without much difficulty. Thus for a prime of the form $4n + 1$, if we denote a residue by R, and a non-residue by N, our formula for the number of sequences RR is $(p + 3)/4$. For the sequence NN we may start from the congruence $xy \equiv N (\text{mod } p)$ and by an exactly similar line of reasoning derive the formula $(p - 1)/4$ for the number of pairs of residues differing by a given non-residue N. Let R' and R'' be two such, so that $R' - R'' \equiv N (\text{mod } p)$. Multiply now this congruence through by the non-residue N' which is such that $NN' \equiv 1 (\text{mod } p)$ and we obtain a pair of non-residues differing by unity. For the sequences NR and RN we may see that they must be equal in number, for since -1 is a residue of primes of the form $4n + 1$ the sequence RN involves the sequence $-R$, $-N$, which when written $p - N$, $p - R$ is seen to be a sequence NR. Now the total number of sequences in the series of numbers $0, 1, 2, 3, 4, \cdots p - 1, p$ is equal to p. Of these there are $(p + 3)/4$ sequences RR, and $(p - 1)/4$, sequences NN. The remaining $(p - 1)/2$ sequences are equally divided among the sequences NR and RN, and are therefore $(p - 1)/4$ in number. With the notation used in stating M. Aladov's results we have, including the residue zero,

$$x = x' = y = \frac{p-1}{4}; \quad \text{and} \quad y' = \frac{p+3}{4}.$$

The case where p is of the form $4n + 3$ may be discussed in the same way,

$$x' = y' = y = \frac{p+1}{4}, \quad \text{and} \quad x = \frac{p-3}{4}.$$

I do not know whether M. Aladov has extended these results to apply to composite moduli or not. This we will do, and first we take the case where the modulus is a power of an odd prime p. We shall show how from a pair of consecutive residues for the modulus p^α we may derive p pairs of consecutive residues for the modulus $p^{\alpha+1}$, unless one of the pair of consecutive residues for p^α is congruent to zero mod p^α.

Let $x_1^2 - y_1^2 \equiv 1 \pmod{p^\alpha}$ yield a pair of consecutive residues. Then $(x_1 + kp^\alpha)^2 - (y_1 + lp^\alpha)^2 \equiv 1 \pmod{p^\alpha}$ will yield the same pair for all values of k and 1. Multiply out and put $x_1^2 - y_1^2 - 1 = mp^\alpha$, then

$$mp^\alpha + 2(kx_1 - ly_1)p^\alpha + (k^2 - l^2)p^{2\alpha} \equiv 0 \pmod{p^\alpha}.$$

The last term on the left is congruent to zero mod $p^{\alpha+1}$ if α is greater than unity. The other terms will also be divisible by $p^{\alpha+1}$ if

$$m + 2(kx_1 - ly_1) \equiv 0 \pmod{p}.$$

In this congruence m, x_1, and y_1 are known and k and l are unknown. It may be shown that a congruence of the form $ax + by + c \equiv 0 (\mathrm{mod}\ p)$ has exactly p roots, unless a and b are divisible by p, and c is not. We defer the proof of this theorem and note that if $y_1 \equiv 0 (\mathrm{mod}\ p^\alpha)$ then the congruence $m+2kx_1 \equiv 0 (\mathrm{mod}\ p)$ yields only one effective pair, for no matter what the value of 1, the expression $(y_1 + lp^\alpha)^2$ will be divisible by $p^{\alpha+1}$ for α greater than one. Thus a pair of consecutive residues for the modulus p^α yields in general p consecutive residues for the modulus $p^{\alpha+1}$ except the consecutive pair (1, 0) which yields the one pair (1, 0). Also if p is of the form $4n + 1$ then a pair $(0, -1)$ is obtainable which likewise yields the single pair $(0, -1)$ for the modulus $p^{\alpha+1}$. The theorem is therefore proved.

The theorem regarding the number of solutions of the congruence $ax + by + c \equiv 0 (\mathrm{mod}\ p)$ is a special case of the more general theorem:

The congruence $ax + by + c \equiv 0 (\mathrm{mod}\ m)$ has $m\delta$ solutions, or none at all according as δ, the greatest common divisor of a, b and m, does or does not divide c.

The congruence clearly has no solutions when δ does not divide c. Take first the case when δ is unity, and let δ' be the greatest common divisor of a and m. Then from the well-known theory of the congruence $ax + b \equiv 0 (\mathrm{mod}\ m)$, it is seen that there will be δ' values of x for every value of y satisfying the congruence $by + c \equiv 0 (\mathrm{mod}\ \delta')$. Now b is prime to δ' and therefore this last congruence has one and only one root β, say. The m/δ' numbers less than or equal to m and congruent to β (mod δ') will serve as values of y, each of which will furnish δ' corresponding values of x. Other values of y will give no values of x at all. The total number of solutions in this case $m/\delta' \cdot \delta'$, or m. If now δ is not unity, we may divide both sides of the congruence through by δ and get the congruence $(a/\delta)x + (b/\delta)y + (c/\delta) \equiv 0 \ (\mathrm{mod}\ (m/\delta))$, which has by the above discussion, m/δ solutions. Let (α, β) be one of these. Then in place of α we may take any one of the δ numbers congruent to α mod (m/δ) which are less than or equal to m. Similarly for β. The total number of solutions is therefore $(m/\delta) \cdot \delta \cdot \delta$, or $m\delta$, which proves the theorem. A similar line of reasoning will show that the number of solutions of the congruence $ax + by + cz + d \equiv 0 \ (\mathrm{mod}\ m)$, is $m^2\delta$, or zero, according as δ, the greatest common divisor of a, b, c, and m does or does not divide d. Also in general, by an easy induction, we may establish that the number of solutions of the congruence $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \cdots a_nx_n + a_{n+1} \equiv 0 (\mathrm{mod}\ m)$ is $m^{n-1}\delta$, or zero,

according as δ, the greatest common divisor of $a_1, a_2, a_3, \cdots a_n$ and m does or does not divide a_{n+1}. These theorems are, of course, special cases of the general problem of simultaneous linear congruences discussed by H. J. S. Smith (*Works*, Vol. II, p. 367).

We can now write down formulæ for the number of consecutive residues for a modulus which is the power of any odd prime p. We have seen that the number of consecutives for a prime of the form $4n + 1$ is $(p + 3)/4$, and leaving out the two sequences (0, 1) and $(-1, 0)$ there remain $\{(p + 3)/4\} - 2$ ordinary sequences. For the modulus p^2 this must be multiplied by p and the two end sequences (0, 1) and $(-1, 0)$ added, which gives for the modulus p^2 the total number of consecutive residues as $\{(p - 5)/4\}p + 2$. In general, for the power p^α the number of consecutive residues will be $\{(p - 5)/4\}p^{\alpha-1} + 2$. The formula is wrong for the prime 5, since in the case of that prime $(p + 3)/4$ is 2, and rejecting the two end consecutives would leave zero. The formula would always give the number as 2. This is correct for α equal to 1, or 2, but for α greater than 2 the correct formula is $4 \cdot 5^{\alpha-3} + 2$.

The argument is easily made also for p of the form $4n - 1$ and the formula is $\{(p - 3)/4\}p^{\alpha-1} + 1$, for the number of consecutive residues for the modulus p^α. The prime 3 is an exception. For α equal to 1 or 2 the number is 1. For α greater than 2 the number is $3^{\alpha-3} + 1$.

For a modulus which is a power of 2 the formula reduces to the greatest integer in $(2^{\alpha-5} + 5)/3$, for 2^α, where α is greater than 4. For α less than 4 the number of sequences is 1.

For the general composite modulus it is now possible to compute the number of consecutive residues. In fact the number of such residues for the product of two relatively prime numbers is equal to the product of the numbers for the two factors. Suppose that the two factors are p and q, and that we have found for each a pair of consecutive residues, so that

$$P_1 - P_2 \equiv 1 (\text{mod } p)$$

and

$$Q_1 - Q_2 \equiv 1 (\text{mod } q)$$

and that these come from the squares,

$$x_1^2 - y_1^2 \equiv 1 (\text{mod } p)$$

and

$$x_2^2 - y_2^2 \equiv 1 (\text{mod } q);$$

determine now k and l to satisfy the congruences:

$$x_1 + kp \equiv x_2 (\text{mod } q),$$

$$y_1 + lp \equiv y_2 (\text{mod } q);$$

p being prime to q these will have one and only one solution each. Thus the number of consecutive residues for the modulus pq is at least as great as the

product of the number of those for the modulus p by that for the modulus q. But since a residue of pq must necessarily be a residue of p and of q separately every pair of consecutive residues for pq will also be a pair of consecutive residues for p and q.

As an illustration of the method of finding consecutive residues for a composite modulus take the modulus $253 = 11.23$. For the modulus 11 we have the consecutive residues 3, 4 arising from 5^2 and 2^2. For the modulus 23 we have the residues 1, 2 arising from 1^2 and 5^2. The congruences $2 + 11k \equiv 5(\text{mod } 23)$ and $5 + 11 \cdot l \equiv 1(\text{mod } 23)$ give $k \equiv 17$, and $l \equiv 8(\text{mod } 23)$. Then $2 + 187$, or 189, and $5 + 88$ or 93 must furnish a pair of consecutive residues for 253. It is found that $189^2 \equiv 48$ and $93^2 \equiv 47$. Since further there are 3 consecutive residues for the modulus 11, and 6 for the modulus 23 there will be 18 for the modulus 253.

In conclusion, a word should be said as to the determination of the possibility of the congruence $x^2 \equiv D(\text{mod } m)$ where D is not restricted to be prime to m. In the first place, it is necessary to consider only the case where the modulus is a power of a prime. For if x can be found such that $x^2 - D$ is divisible by pq, where p and q are relatively prime, then $x^2 - D$ will be divisible by p and q, and conversely. Consider then the modulus p^α; and let $D = Ap^\lambda$ where A is prime to p.

If λ is greater than α the congruence will have the root $x \equiv 0$.

I. Suppose that λ is less than α, and suppose first that A is a residue p of and write $A \equiv y^2(\text{mod } p^\alpha)$. Then if λ is even the congruence is possible and has the root $x \equiv \pm yp^{\lambda/2}$.

II. If A is a residue of p and λ is odd the congruence has no roots. For writing $x^2 \equiv Ap^\lambda + Mp^\alpha$ then x contains p^λ as a factor. It must then contain another factor p. But the right side can contain no such factor.

III. If A is a non-residue of p and λ is even the congruence is impossible. For dividing out p^λ we would have the impossible congruence

$$x'^2 \equiv A(\text{mod } p^{\alpha-2}).$$

IV. If A is a non-residue, and λ is odd then Ap^λ is a non-residue. Proof as in case II. In general, then Ap^λ is a residue of p^α when λ is greater than or equal to α and when λ is less than α and even and A a residue of p.

THE GENERAL SOLUTION OF A CLASS OF INFINITE SYSTEMS OF EQUATIONS IN AN INFINITE NUMBER OF VARIABLES.

By THOMAS. E. MASON, Indiana University.

The problem of this paper is to find the general solution of the system of equations

$$(1) \qquad \sum_{j=0}^{\infty} a_{ij}u_j = c_i, \quad i = 0, 1, 2, 3, \cdots,$$

where the coefficients a_{ij} are polynomials in i, of the form

$$a_{ij} = \alpha_j^{(0)} + \alpha_j^{(1)}i + \alpha_j^{(2)}i^2 + \cdots + \alpha_j^{(n)}i^n.$$

The system can be expressed in the form

$$(2) \quad \sum_{j=0}^{\infty}\alpha_j^{(0)}u_j + i\sum_{j=0}^{\infty}\alpha_j^{(1)}u_j + i^2\sum_{j=0}^{\infty}\alpha_j^{(2)}u_j + \cdots + i^n\sum_{j=0}^{\infty}\alpha_j^{(n)}u_j = c_i, \quad i=0, 1, 2, \cdots.$$

A necessary and sufficient condition that the general solution of (2) shall be the general solution of (1) is that for every solution of (1) each of the infinite series indicated by the summations in (2) shall converge. This condition will be shown to be fulfilled and the system will be studied in the form (2).

When $i = 0$ the infinite series in the first term of (2) converges[1] since it is the first equation of (1). Now add the negative of that series to (1) and the result can be written in the form

$$(3) \qquad \sum_{j=0}^{\infty} (\alpha_j^{(1)}i + \alpha_j^{(2)}i^2 + \cdots + \alpha_j^{(n)}i^n)u_j = k_i, \quad i = 0, 1, 2, \cdots.$$

If any n consecutive equations of (3) are multiplied, the first by m_1, the second by $m_2, \cdots$, the nth by m_n, and the products added the result is of the form

$$\sum_{j=0}^{\infty} \{[m_1i + m_2(i+1) + \cdots + m_n(i+n-1)]\alpha_j^{(1)} + [m_1i^2 + m_2(i+1)^2 + \cdots + m_n(i+n-1)^2]\alpha_j^{(2)} + \cdots + [m_1i^n + m_2(i+1)^n + \cdots + m_n(i+n-1)^n]\alpha_j^{(n)}\}u_j = k_i'.$$

Now let t be any integer of the set $1, 2, 3, \cdots, n$. Then for each value of i the m's can be chosen in such way as to make the coefficient of $\alpha_j^{(t)}$ equal to 1 and the coefficients of $\alpha_j^{(k)}$ equal to 0, when $k \neq t$. That the m's can be so chosen is evident from the fact that they are thus restricted to satisfy n non-homogeneous equations in n unknowns, the determinant of the system being different from zero. This would give

$$\sum_{j=0}^{\infty}\alpha_j^{(t)}u_j = k_t'', \quad t = 1, 2, \cdots, n,$$

that is, each of the infinite series of (2) converges for every solution of (1). Hence the general solution of (2) is the general solution of (1) and vice versa.

Write

$$(4) \qquad \sum_{j=0}^{\infty}\alpha_j^{(t)}u_j = B_t, \quad t = 0, 1, 2, \cdots, n;$$

then (2) can be written

$$(5) \qquad B_0 + iB_1 + i^2B_2 + \cdots + i^nB_n = c_i, \quad i = 0, 1, 2, \cdots.$$

Difference this equation $n + 1$ times with respect to i and the result is

[1] Only such values of u_j are called a solution as make every first member of (1) convergent.

(6) $$\Delta^{n+1} c_i = 0,$$
that is,

$$c_{i+n+1} - (n+1)c_{i+n} + \frac{(n+1)n}{2!} c_{i+n-1} - \cdots + (-1)^{n+1} c_i = 0.$$

This is a necessary condition on the c's that equations (2) shall be consistent. The same condition on the c's can be obtained by taking consecutive values of i and eliminating the B's. We suppose that the c's satisfy this condition.

If $n + 1$ consecutive values of i are taken in (5) the B's can be found in terms of the c's, since the determinant of the $n + 1$ equations is different from zero. If these values of B_t are substituted in (4) the problem is reduced to solving the finite set of equations in an infinite number of variables:

(7) $$\sum_{j=0}^{\infty} \alpha_j{}^{(t)} u_j = B_t, \quad t = 0, 1, 2, \cdots, n.$$

If possible select $n + 1$ of the u's such that the determinant of their coefficients is different from zero. Transpose to the second member of the equations the remaining u's. These can be given arbitrary values, provided that the second member of each equation converges. Then the remaining $n + 1$ u's can be found by the Cramer method. The solution thus obtained has an infinite number of arbitrary elements. Obviously it is the general solution.

If no $(n + 1)$th order determinant, not zero, exists in the first member of (7) it will be shown below that the equations in the system (7) are linearly dependent. If the equations are linearly dependent and are consistent there will be conditions on the B's like

$$\sum_{t=0}^{n} b_t B_t = 0,$$

where the b's are constants. The number of such conditions will depend on the number of equations which are dependent on the others. Since the B's are expressed in terms of the c's these are additional conditions which must be fulfilled by the c's.

To show that if all the $(n + 1)$th order determinants of the coefficients in (7) are zero the equations are linearly dependent, it will be shown that if they are linearly independent there is at least one $(n + 1)$th order determinant different from zero.

Set up the matrix of the coefficients of (7)

$$\left\| \begin{array}{ccccc} \alpha_0{}^{(0)} & \alpha_1{}^{(0)} & \alpha_2{}^{(0)} & \alpha_3{}^{(0)} & \cdots \\ \alpha_0{}^{(1)} & \alpha_1{}^{(1)} & \alpha_2{}^{(1)} & \alpha_3{}^{(1)} & \cdots \\ \alpha_0{}^{(2)} & \alpha_1{}^{(2)} & \alpha_2{}^{(2)} & \alpha_3{}^{(2)} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_0{}^{(n-1)} & \alpha_1{}^{(n-1)} & \alpha_2{}^{(n-1)} & \alpha_3{}^{(n-1)} & \cdots \\ \alpha_0{}^{(n)} & \alpha_1{}^{(n)} & \alpha_2{}^{(n)} & \alpha_3{}^{(n)} & \cdots \end{array} \right\|$$

The equations being assumed linearly independent, no row has each term zero, nor can any row be made to have each term zero by addition of multiples of other rows. If $\alpha_0{}^{(0)} = 0$ then we can move to the position of first column a column where the term in the first row is different from zero since not all the terms of any row are zero. Then by addition of multiples of this row all but the first term of the first column can be made zero. If $\alpha_1{}^{(1)} = 0$ then some column following has the term in the second row different from zero and it can be brought to occupy the position of second column. As before all the terms of this column below the second can be made zero by addition of multiples of the second row. Proceeding in this way a new matrix is obtained the first $(n + 1)$th order determinant of which will have all the terms below the principal diagonal equal to zero and each term of that diagonal different from zero. The determinant of the original matrix composed of those $n + 1$ columns which form the first $(n + 1)$th order determinant of the transformed matrix, is obviously not zero. Hence if the equations are linearly independent there will be at least one $(n + 1)$th order determinant different from zero, and hence if every $(n + 1)$th order determinant of the coefficients in (7) is zero the equations are linearly dependent.

In the case of linear dependence the equations which are dependent on others can be omitted and the general solution can be found from the linearly independent equations. This can be done as above by making arbitrary, with restriction as to convergence, all the u's except as many as there are equations and solving for these by the ordinary method.

PRECISE MEASUREMENTS WITH A STEEL TAPE OR WIRE.

By GEORGE R. DEAN, Rolla, Mo.

In the measurement of horizontal distances with a steel tape or wire in cases where all attainable accuracy is required, there will be considerable calculation avoided if the engineer has the means of knowing how to adjust the tension so that the elongation due to tension may balance the correction for sag or droop. Under ordinary conditions the correction for temperature is negligible, and when not negligible is easily applied and need not be discussed here.

We will derive a formula for the sag correction and the stretch separately. Then by equating these corrections, derive a formula for the tension in terms of the weight, cross-section and modulus of elasticity of the tape or wire.

Correction for Sag.—This is usually derived by the aid of the calculus in text-books on mechanics, but the calculus is unnecessary for this purpose, if we assume that the curve is the arc of a circle. The approximate formula derived in this way is the same as that obtained by integration when the curve is taken as a catenary or a parabola.

Let R = radius of circle,
l = length of tape or wire in inches,
α = number of radians in half the angle at center,
h = horizontal distance in inches,
δ = sag in inches.

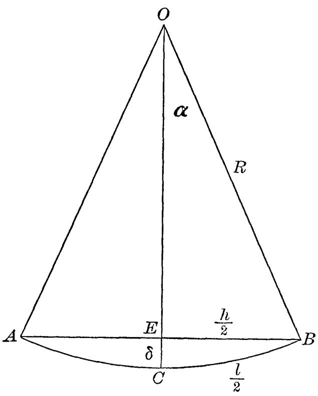

FIG. 1.

In Fig. 1,

$$OA = OB = OC = R,$$

$$\angle\ COB = \alpha, \quad \text{and} \quad EC = \delta.$$

$$AB = h, \quad ACB = l.$$

Then

$$l = 2R\alpha, \tag{1}$$

$$h = 2R \sin \alpha, \tag{2}$$

$$\delta = R(1 - \cos \alpha). \tag{3}$$

Since α is in practice a very small angle, we may write as approximations,

$$\sin \alpha = \alpha - \frac{\alpha^3}{6}, \quad \text{and} \quad \cos \alpha = 1 - \frac{\alpha^2}{2},$$

using the first two terms of the sine and cosine series. Then our equations (1), (2), (3), become

$$l = 2R\alpha, \tag{4}$$

$$h = 2R\left(\alpha - \frac{\alpha^3}{6}\right), \tag{5}$$

$$4\delta = 2R\alpha^2. \tag{6}$$

Dividing (6) by (4),

$$\alpha = \frac{4\delta}{l}. \tag{7}$$

Dividing (5) by (4),

$$1 - \frac{\alpha^2}{6} = \frac{h}{l}. \tag{8}$$

From (8),

$$h = l - \frac{l\alpha^2}{6}.$$

Substituting value of α from (7)

$$h = l - \frac{8\delta^2}{3l}. \tag{9}$$

Then the correction for sag is $8\delta^2/3l$.

Correction for Tension.—Let T be the pull at each end, w the weight of tape or wire per unit length, a the area of cross-section in square inches, E the modulus of elasticity of the material, θ the angle between the direction of the pull and the horizontal. The smaller the sag the more nearly will CF be equal to CE (Fig. 2).

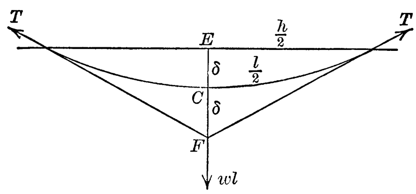

FIG. 2.

The forces T, T and wl are in equilibrium. The vertical component of each pull is $T \sin \theta$. Then we have

$$2T \sin \theta = wl. \tag{10}$$

Now

$$\sin \theta = \frac{2\delta}{\frac{2}{l}} = \frac{4\delta}{l}, \text{ approximately.}$$

Then

$$2T\left(\frac{4\delta}{l}\right) = wl,$$

or

$$T = \frac{wl^2}{8\delta}. \tag{11}$$

If x denote the elongation,

$$x : l = \frac{T}{a} : E,$$

or

$$x = \frac{lT}{aE}. \tag{12}$$

Equating this to the correction for sag,

$$\frac{lT}{aE} = \frac{8\delta^2}{3l}. \tag{13}$$

From (11),

$$\delta = \frac{wl^2}{8T}.$$

Substituting in (13),

$$\frac{lT}{aE} = \frac{8w^2l^4}{64T^2 \cdot 3l} = \frac{8w^2l^4}{192T^2l}.$$

Solving for T,

$$T^3 = \frac{aEw^2l^2}{24}, \tag{14}$$

or

$$T = \sqrt[3]{\frac{aEw^2l^2}{24}},$$

and since $wl = W$ the weight of tape,

$$T = \sqrt[3]{\frac{aEW^2}{24}}. \tag{15}$$

Numerical Example.—If the weight of tape is 5 pounds, the modulus of elasticity 30,000,000 pounds per square inch, and the area of cross-section .015 square inch,

$$T = \sqrt[3]{\frac{.015 \times 30{,}000{,}000 \times 25}{24}} = 77.5 \text{ pounds.}$$

and

$$\delta = \frac{Wl}{8T} = \frac{5 \times 5}{8 \times 77.5 \times .015 \times 0.3} = \frac{25}{77.5 \times .036} = 8.9 \text{ inches.}$$

Here

$$l = \frac{5}{.015 \times 0.3} = 1{,}111.1 \text{ inches} = 92.5 \text{ feet.}$$

BOOK REVIEWS.

W. H. BUSSEY, Chairman of the Committee.

Practical Geometry and Graphics. A text book for students in technical and trade schools, evening classes, and for engineers, artisans, draughtsmen, architects, etc. By E. L. Bates and F. Charlesworth. Van Nostrand, New York, 1912. viii + 621 pages. \$2.00.

Practical Mathematics. By E. L. Bates and F. Charlesworth. Van Nostrand, New York, 1912. viii + 513 pages. \$1.50.

These two texts, which have been written in part to prepare for the examinations of the London Board of Education, are essentially what they claim to be, that is for practical men. In fact they almost take the form of an engineering pocket book. The ground covered is enormous, the proofs whenever given are brief and frequently of an experimental nature, such as would appeal to good

mechanics with little mathematical training. Both texts are very well illustrated, well written, and published in an attractive form, with an immense number of examples which make them very convenient for students working alone.

The two books cover a great deal of common ground. The first is written more especially for draughtsmen and for such practical work as requires graphics. It deals with conics, graphostatics and descriptive geometry. The second book has a broader scope and is more especially for mechanics and designers. It contains algebra, geometry, trigonometry, vectors, and calculus. Both books are to be recommended to the class of students which they aim to reach, and also to more advanced readers for the numerous and well chosen practical examples which they contain.

S. LEFSCHETZ.

Life Assurance Primer. A text book dealing with the practice and mathematics of life insurance, for advanced schools, colleges and universities. By HENRY MOIR. Third edition, revised and enlarged. The Spectator Company, New York, 1912. vii + 230 pages. $2.00.

In recent years, there has been a tendency both in Europe and America to extend university education to include something of the principles of insurance. This work in insurance has often taken the form of a treatment of the economics of insurance; but it has also been recognized that any substantial development of the elementary principles of insurance very naturally takes a mathematical form. The present book is found to be very useful as a beginning text for teaching the elements of this subject. The reviewer has used the second edition, and is at present using the third edition in a class of twenty students of junior and senior rank, and is prepared to say that he finds the book teachable. In this last edition the material has been thoroughly revised, and much improved for purposes of university instruction. A new chapter has been added dealing with the organization and management of a life company. One of the new features is the introduction of questions, exercises, and problems at the end of each chapter. This addition will surely commend itself to teachers of the subject. The book serves well as a text for a beginning course of approximately two hours per week for a semester; and, although the reading of the book requires hardly any more mathematics than the algebra usually included in a freshman course, it is not too easy for the beginner, as a comprehension of the business situations involved requires considerable knowledge, and the expression of such situations in mathematical form requires a good deal of thought.

H. L. RIETZ.

Source Book of Problems for Geometry. By MABEL SYKES, with the coöperation of H. E. SLAUGHT and N. J. LENNES. Allyn and Bacon, Boston, 1912. viii + 372 pages.

The present volume is a welcome contribution to the endeavor to make the mathematical work in our schools more practical and tangible. The belief

that mathematics is an excellent discipline has for many years removed the mathematics of the secondary schools farther and farther from the daily life and experience of the pupils. The text-books have emphasized the logical side, and have contained a large number of exercises and problems for drill and a few problems in applied mathematics which are for the most part spurious. We are, however, coming to realize in a larger measure that the great majority of boys and girls finish their education in the grammar school or the high school, and that the algebra and the geometry of the schools should prepare pupils as far as possible to do well the work that falls to them after leaving school.

In courses in architecture and design this *Source Book of Problems* meets the demand that a text-book shall have continuous and practical connection with working conditions. All of the problems have to do with real things, and are solved by the practical application of geometrical principles. The titles of the six chapters show the practical character of the book. The titles are: (1) Tile designs; (2) Parquet floor designs; (3) Miscellaneous industries; (4) Gothic tracery: Forms in circles; (5) Gothic tracery: Pointed forms; (6) Trusses and arches. The index of theorems and problems will aid the teacher in selecting the material so as to keep the logical continuity intact. In solving the problems the principles of arithmetic, algebra, and geometry are combined to such an extent that a thorough review of the former subjects is given, while the use of algebraic notation and methods makes the solutions much simpler and easier.

As a supplementary book in the high school course in geometry this text will, in the writer's opinion, prove of great value. Geometrical forms occur so frequently in tiles, mosaics, needlework, jewelry, iron grills, steel ceilings, tracery of windows and other architectural forms that a study of these forms will add greatly to the interest of geometry. Moreover it will be a pleasure to recognize known geometrical forms in architectural details which meet the eyes daily but have failed to attract attention.

The historical accounts of the various designs and the pictures of windows and other architectural details of famous buildings are of general educational value. There are over 1,800 exercises, and many of these are general problems from which it is possible to form any number of numerical problems. The drawings and illustrations number more than 450; they are well drawn and reproduced on excellent paper. The extensive bibliography and list of commercial catalogues enables the teacher to secure additional material if desired. The author, editors and publishers are to be congratulated on the appearance of this book.

H. E. Cobb.

PROBLEMS AND QUESTIONS.

B. F. Finkel, Chairman of the Committee.

The solutions of problems and the proposal of new problems will be continued in the June issue.

NOTES AND NEWS.

Florian Cajori, Chairman of the Committee.

The answer book to Davis's Calculus is not expected to be published until next fall.

A new "College Algebra" by W. Benjamin Fite, Professor of Mathematics in Columbia University, has just been published by D. C: Heath and Company.

The meeting of the Association of Ohio Teachers of Mathematics and Science which was to have been held at Columbus at the end of March, was postponed indefinitely on account of the floods over Ohio.

Dr. Willy Wien, professor of physics in the University of Würzburg, delivered in April at Columbia University six lectures on "Recent Problems of Theoretical Physics."

It is proposed to hold an international congress of education in connection with the Panama-Pacific International Exposition, which is to take place in San Francisco in 1915. Efforts are being made to hold a large number of other scientific meetings in connection with this exposition.

M. d'Ocagne, professor at the Paris Polytéchnic School, was elected president of the "Commission du Rêpertoire" to succeed M. Poincaré. This commission is publishing an extensive catalogue, in the form of small cards, of the mathematical literature up to 1900. These cards are sold by the well-known French publishers Gauthier-Villars, Paris, France.

The Smithsonian Institution of Washington obtained permission to include in the "Annual Report of the Board of Regents to Congress" the paper by Professor G. A. Miller, entitled *Some thoughts on modern mathematical research,* which appeared in Science, June 7, 1912.

During the latter part of April, the National Academy of Sciences celebrated its fiftieth anniversary at the Smithsonian Institution of Washington. A number of distinguished scientists were invited as guests. Among the ten new members elected at this meeting was Dr. L. E. Dickson, professor of mathematics at the University of Chicago.

The list of papers presented at the April meeting of the California section of the American Mathematical Society was as follows: "A set of postulates for the algebra of positive rational numbers with zero," by Mr. B. A. Bernstein; "On

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INCENTIVES TO MATHEMATICAL ACTIVITY.

By H. E. SLAUGHT, University of Chicago.

Sources. During the past ten years there has been a marked activity among teachers of mathematics in this country, looking toward improvement in methods and the reconsideration of subject matter, especially with reference to selection and emphasis. In this activity we are by no means alone. In fact, there has probably been still greater readjustment on the other side of the water. Doubtless the so-called Perry movement in England may be considered as one of the chief awakening influences underlying our own activity. But Professor Moore's presidential address[1] before the American Mathematical Society in December, 1902, "On the Foundations of Mathematics," may have been the real source of inspiration in this country which led to the special period of activity beginning at about that time.

The International Commission. As a world movement, all the activities in the various civilized countries were finally centered in the International Commission on the Teaching of Mathematics appointed by the Fourth International Congress of Mathematicians at Rome in 1908, on the motion of Professor D. E. Smith, of Columbia University. For four years this Commission occupied itself with collecting and publishing the fullest possible information as to the present status of the teaching of mathematics in each of the countries represented, not making any attempt thus far to propose or discuss reforms of any kind. The first report of this commission was made at the Fifth International Congress[2] held in Cambridge, England, in August 1912, where twenty-seven countries were represented and one hundred and fifty printed reports were presented, leaving about fifty still to be completed. The work of the commission will be continued

[1] *Bulletin of the American Mathematical Society*, Vol. IX, May, 1903, pp. 402–424. This article contains numerous references to the literature on this subject.

[2] For a general report of this Congress see the *Bulletin of the American Mathematical Society*, Vol. XIX, December, 1912, January, 1913. For a special report on the pedagogical aspects of the work of this Congress see THE AMERICAN MATHEMATICAL MONTHLY, Vol. XIX, October, 1912.

for another four years, giving opportunity for the study of numerous other topics not yet touched, and especially for the analysis and coordination of the information contained in the reports already published, for instance, a summary of the large features of these reports which could be printed in form for general distribution. The reports for the United States have been published by the Bureau of Education at Washington and may be had upon application to the Commissioner of Education.

Other Organizations. The work of the International Commission, however, must be considered as the outgrowth of activities which have their underlying causes far beneath the surface. For instance, in this country during the ten year period above referred to, there have been organized practically all of the now existing associations[1] of teachers of mathematics, including many city and state organizations and three large bodies,—in New England, the Middle States, and the Central West respectively. Two journals of wide circulation, supported by these associations, have also been founded during this same period; namely, *School Science and Mathematics* published at Chicago and *The Mathematics Teacher* published at Syracuse, N. Y.

All these organizations have been keenly active, especially the three large associations. For the most part, they have been engaged in the study of existing conditions and of means of improvement, through the reports of committees based upon long and serious investigations. Among the more important of such reports have been; a geometry syllabus by the New England Association, an algebra outline by the Middle States Association, another by the Missouri State Association, outlines for both algebra and geometry by a state conference meeting annually at the University of Illinois, a syllabus of algebra by the Central Association which was printed and reprinted, to meet the demands for it, until its circulation reached more than 20,000, and a geometry syllabus and an outline of a course in unified mathematics also by the Central Association, both of which were widely distributed and studied. Without doubt all of these documents have stimulated wide interest in pedagogical questions relating to secondary mathematics, and one, at least, has produced a profound impression upon the selection and presentation of subject matter as reflected in the newer text-books, namely, the algebra report of the Central Association.

The National Federation. Another organization in the interest of national coöperation among associations of teachers of mathematics and allied sciences was undertaken four years ago, namely, the Federation of Teachers of the Mathematical and Natural Sciences, which includes a membership of about 2,500 acting through the associations above enumerated. It was the purpose of this federation to deal with questions of national scope and it has been effective in at least two important matters, namely, (1) a bibliography of science terms and (2) a national geometry syllabus by a committee of fifteen, seven representing universities and eight representing secondary schools, acting under the joint auspices of the Federation and the National Education Association. This

[1] For a partial list of these associations see *School Science and Mathematics* for March, 1913.

geometry report of eighty pages was published in full in *School Science and Mathematics* in 1911, and again in final form in the *Mathematics Teacher* in 1912, and also in the proceedings of the National Education Association for 1911 and 1912. It was distributed through the Bureau of Education in Washington and has had a circulation, in this country and abroad, of 27,000 copies, and a request has just been granted to the Irish Journal of Education for its republication there in full.

The Underlying Causes. This decade of remarkable activity on the part of the teachers of mathematics in this country can be accounted for only on the hypothesis of a widespread and deep-seated feeling of unrest and dissatisfaction with the present methods and results, a conviction that in some way the mathematics teaching in the schools fails to connect vitally with the preparation for living in the twentieth century needed by the vast majority of the pupils in the schools. Possibly the following quotation from Professor Moore's address, above cited, may be the true analysis of the situation:

"He (Perry) asserts as essential that the boy should be *familiar* (by way of experiment, illustration, measurement, and by every possible means) with the ideas to which he applies this logic; and moreover that he should be thoroughly *interested* in the subject studied. . . .

"As a pure mathematician, I hold as the most important suggestion of the English movement the suggestion of Perry's just cited, that by emphasizing steadily the practical sides of mathematics, that is, arithmetic computations, mechanical drawing and graphical methods in general, in continuous relations with problems of physics and chemistry and engineering, it would be possible to give very young students a great body of the essential notions of trigonometry, analytic geometry, and the calculus. This is accomplished on the one hand by the increase of attention and comprehension obtained by connecting the abstract mathematics with the subjects which are naturally of interest to the boy, so that, for instance, all the results obtained by theoretic process are capable of check by laboratory process, and on the other hand by a diminution of emphasis on the systematic and formal sides of the instruction in mathematics. Undoubtedly many mathematicians will feel that this decrease of emphasis will result in much if not irreparable injury to the interests of mathematics. But I am inclined to think that the mathematician with the catholic attitude of an adherent of science in general (and at any rate with respect to the problems of the pedagogy of elementary mathematics in particular there is no other rational attitude) will see that the boy will be learning to make practical use in his scientific investigations, to be sure in a naïve and elementary way, of the finest mathematical tools which the centuries have forged, that under skillful guidance he will learn to be interested not merely in the achievements of the tools but in the theory of the tools themselves, and that thus he will ultimately have a feeling toward his mathematics extremely different from that which is now met with only too frequently—a feeling that mathematics is indeed itself a fundamental reality of the domain of thought, and not merely a matter of symbols and arbitrary rules and conventions."

Whether or not this is the true interpretation of the present conditions, it is certainly significant that Professor Klein advocates practically the same remedy and that this is one of the questions which the International Commission has set for itself in its further deliberations.

The Indirect Results. Whatever may be the true causes of this widespread agitation and discussion, and whatever may be the final conclusions reached, this much is certain, that teachers of mathematics, the world over, are doing a vast amount of reading, investigating, experimenting, and careful thinking about the problems that confront them, and in this very attitude of mind lies the hope and the assurance that ultimately sane conclusions will be reached. In this country, especially, where there is a minimum of centralized authority, the final

disposition of these important questions depends largely upon such a development of thoughtful attention and investigation on the part of the general body of teachers as seems to be now in progress.

The indirect result, therefore, of all these discussions has been to stir teachers out of their lethargy and to encourage independent study outside of their routine duties. This seems to be the most plausible explanation of the vast increase in attendance upon the summer sessions at all the institutions where these are held. The report upon this topic[1] by the present writer, as a part of the American report of the International Commission, showed that the development of summer schools (that is, of summer sessions of normal schools, colleges and universities, conducted on the same basis as the work in any other part of the year), has been a remarkable phenomenon, and this has gone forward by leaps and bounds ever since that report was made. Along with the rapid increase of summer attendance in general, and of students of mathematics in particular, has gone the enrichment of the curricula, especially in the number and variety of courses which may count toward higher degrees, and in the courses on the history and teaching of mathematics. All this cannot fail to produce a profound impression upon those who are availing themselves of these opportunities, to arouse new enthusiasm, to stimulate fresh activity, to develop intelligent reaction on all the questions which confront the teachers of mathematics in this country.

The College Situation. For the most part, the foregoing data and observations apply to the teaching of mathematics in the secondary schools. All of the associations mentioned are primarily for secondary teachers, although many of the most enthusiastic members are college teachers. It is true that the investigations of the International Commission include schools of all grades from the elementary schools to the universities, but so far almost no attention has been given to the teaching of mathematics in the colleges, although it will readily be admitted that the problem is not so very different there, especially in first two years. It is practically certain that every charge that has been made against the teaching of mathematics in the secondary schools may be made with equal force in regard to the Freshman and Sophomore courses in college. In fact, there may be wider room for desirable modification in such subjects as analytic geometry and the calculus than in elementary algebra and geometry, if we may judge by some of the more recent texts and by the present practice of not a few colleges and universities.

But how shall the rank and file of teachers of college mathematics learn of the modifications in progress and of the success or failure of experiments in other institutions. Clearly the only way is by intercommunication, which must be either personal and hence very limited, or through some general medium of publication. It is precisely in this capacity that the editors of the MONTHLY believe its usefulness should be developed, and they are ready to devote any reasonable amount of space to the discussion of questions of this character.

[1] Published in the AMERICAN MATHEMATICAL MONTHLY, Vol. XIX, July, 1912. Afterwards also printed with the other reports in the Government bulletins.

[2] For one important exception see an item in the Notes and News of this issue.

However, it would not seem wise that such a journal should be devoted entirely or chiefly to pedagogical discussions as such. It may, and often does, happen that the stimulus derived from reading an article not directly pedagogical in its nature may produce a reaction strongly beneficial on the pedagogical side. Such, for instance, may be the effect of articles like those of Professors Hedrick, Huntington and Coolidge in the present issue, and such without doubt has been the effect of Professor Cajori's "History of Logarithms" which has been running since January. Even those articles which appear too abstruse for a given reader may provide just the stimulus which he particularly needs, not only to insure his continued mental activity but to actually furnish a form of mental stimulus which may be essential to his successful work as a teacher. This is the reason for publishing in the MONTHLY such articles as those of Professor Lehmer in the May issue, of Dr. Miles in the April issue, and of Professor Dickson in the March issue. (See the reference to this question under Notes and News.)

Conclusion. While it is true that agitation does not necessarily mean progress, it is also true that there is seldom any progress without agitation. We confidently believe that the unprecedented activity among teachers of mathematics during the past decade has resulted, and will further result, in substantial progress. It should be a keen incentive to every teacher that he or she may have some active part in our nation-wide determination to reconsider the foundations of our teaching and to improve our methods wherever possible; and that in this effort we are allying ourselves with a world-wide movement toward the same end.

HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

BY FLORIAN CAJORI, Colorado College.

V. GENERALIZATIONS AND REFINEMENTS EFFECTED DURING THE NINETEENTH CENTURY.

THE GENERAL POWER AND LOGARITHM.

We have seen that the general theory of a^b, where both a and b are complex numbers, was outlined by L. Euler in his *Recherches sur les racines imaginaries des équations* of 1749, and that this paper failed to command the attention of mathematicians. We shall see now that three quarters of a century later the theory of the general power was elaborated by mathematicians of Germany, England, France and the Netherlands. At the opening of the nineteenth century this subject appeared difficult to many, as may be inferred from a paper of A. Q. Buée, in which the author contends that $\sqrt{-1}$ signifies perpendicularity,[1] and is finally led to the conclusion that $(\sqrt{-1})^n = \pm n(\sqrt{-1})$. The difficulty of the subject appears also in a paper of Argand who in 1813 ventured the statement that the expression $(\sqrt{-1})^{\sqrt{-1}}$ "would offer the simplest example of a

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[1] *Philosoph. Trans.* for the year 1806, London, 1806, p. 67.

quantity irreducible to" the form $a + ib$.[1] Exception was taken to this statement by J. F. Français, professor in Metz, who pointed out that Euler had found $(\sqrt{-1})^{\sqrt{-1}} = e^{-1/2\pi}$, and showed that $(c\sqrt{-1})^{d\sqrt{-2}}$ is reducible[2] to $a + ib$. In the course of the next fifteen or twenty years sufficient familiarity was acquired with imaginaries to enable several mathematicians to grapple successfully with the theory of the general power. In the early history of the logarithms of positive numbers it was found surprising that logarithms were invented independently of exponents. Now another surprise is in store, namely that the theory of the general power is made to depend upon the theory of logarithms. All interpretations of a^b, where a and b are complex numbers, involve previously established results on logarithms. It thus appears that, historically, the logarithmic concept is the more primitive.

From the subtelty of general logarithmic theory arises the danger of its occasional recrudescence. We give an instance of this before we proceed to the general theory. Traces of confusion are found in an article by A. J. H. Vincent, professor at the Royal College at Reims—an article which possesses many points of merit. It touches upon the multiple nature of the logarithmic curve and its association with logarithmic theory. It is entitled, *Considérations nouvelles sur la nature des courbes logarithmiques et exponentielles*.[3] The author lets the variable x in $y = a^x$ take fractional values with even denominators, and obtains the usual two real branches. Vincent's article is discussed by George Salmon in his *Higher Plane Curves* (1879, page 286). Salmon accepts Vincent's idea that the curve has multiple branches. Vincent defines $1/n$ as the logarithm of all the numbers $a^{1/n}$, and concludes that, if a is positive, some $-$ numbers have real logarithms; if a is negative, some $+$ and some $-$ numbers have no logarithms, and so on. Vincent studies the discontinuities of his curves, but does not inquire into the consistency of his multiple-definition of logarithms. Peacock expresses himself as follows:

"The question of the identity of the logarithms of the same number, whether positive or negative, . . . has been frequently resumed in later times. The arguments in favour of the affirmative of this proposition, which were for the most part founded upon the analytical interpretation of the properties of the hyperbola and logarithmic curve, were not entitled to much consideration, in as much as they were not drawn from an analysis of the course followed in the derivation of the symbolical expressions themselves and from the principles of interpretation which those laws of derivation authorized."[4]

The Vincent article was criticized by J. P. W. Stein,[5] who took the view that every fractional index may be converted into one having an even denominator so that there would be a double number corresponding to *every* logarithm. Vincent was criticized also by D. F. Gregory who said:[6]

"It might, perhaps, have weakened his belief in the correctness of the results, if he had come to the conclusion, as he ought to have done, that the same logarithm corresponded to positive, negative, and impossible quantities. These last he seems quite to have overlooked."

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[1] Gergonne, *Annales de Math.*, T. IV., 133–147.
[2] Same journal, Vol. IV, pp. 71–73.
[3] *Gergonne's Annales de math.*, Nismes, 1824 et 1825, T. XV., pp. 1–38.
[4] *Report of British Ass'n*, London, 1834, p. 266.
[5] Gergonne, *Annales de math.* T. XV, p. 231.
[6] *Cambridge Math. Jour.*, Vol. I, 1837, "On the Impossible Logarithms of Quantities."

A more serious recrudescence is seen in an article by L. C. Bouvier,[1] ex-officier de génie. In 1823 he derived the eulerian formulæ for $\log x$ and $\log(-x)$ from the relation $\log x = n(\sqrt[n]{x} - 1)$, $n = \infty$. Then, solving for x, he got $x = \left(1 + \frac{1}{n}\log x\right)^n$, *d'où, à cause de n infini, on peut conclure* $x = \left(1 + \frac{1}{n}\log x\right)^{n+1/k}$ $= \left(1 + \frac{1}{n}\log x\right)^n \cdot \left(1 + \frac{1}{n}\log x\right)^{1/k} = N\sqrt[k]{1}$, N being real and k any positive integer. Thus, not only is the logarithm of a number many-valued, but for any one logarithm the anti-logarithm is many-valued. For $k = 2$, $\log x = \log(-x)$. *Ces considérations nous semblent de nature à terminer, une fois pour toutes, le différend qui s'est élevé autrefois entre Euler et d'Alembert, sur la nature des logarithmes des quantités négatives.* Bouvier made no attempt to test the inherent consistency of his system. That there was inconsistency was claimed by J. P. W. Stein,[2] professor at the gymnasium at Trèves. Stein pointed out that Bouvier could not accept the eulerian formulæ, $\log x = r + 2m\pi\sqrt{-1}$, ($x$ positive) and $\log x = r + (2m+1)\pi\sqrt{-1}$ (x negative), and at the same time take $x = N\sqrt[k]{1}$. Starting into an untrodden path, Stein advances new arguments to show that Euler's formulæ are incorrect. From the relation $e^{r+z\sqrt{-1}} = y$, r being the real logarithm of y, z a number to be determined, Stein first deduces $e^{z\sqrt{-1}} = 1$ and from this obtains Euler's formulæ. *Mais il faut observer que, si* e^r *peut admettre plusieurs valeurs différentes*, say k values, then $e^r = y[\cos(2p\pi/k) + \sqrt{-1}\sin(2p\pi/k)]$. This result, combined with the eulerian values, yielded him $\log y = r + [2m - (2p/k)]\pi\sqrt{-1}$, $\log(-y) = r + [2m' + 1 - (2p'/k)]\pi\sqrt{-1}$. When k is even, he obtained $\log y = \log(-y)$. We see that, in Stein's system, the relation $x = \log y$ is no longer defined by the simple equation $e^x = y$, but by the equation $e^{x-(2p\pi/k)} = y$. The general expression for the logarithm of a number involves two independent arbitrary constants, m and p, instead of one only, as in the eulerian system. More perplexing is the circumstance that the number of values k which e^r takes, was seemingly not supposed to be the same in all cases or for all numbers. The author does not explain how, under these conditions, his system can yield a theory of logarithms that is general, consistent and useful. Certainly Stein did not move in the direction of Cauchy, who, as we shall see, by the introduction of principal values, aimed at greater simplicity and the inauguration of law and order.

Proceeding to the general theory of a^b, we observe that an author of very marked influence in the development of this theory was Martin Ohm, professor in Berlin. In 1811, when he was privat-docent at the university of Erlangen, he conceived the idea of writing the system of mathematics which appeared later under the title *Versuch eines vollkommen consequenten Systems der Mathematik*, Nürnberg, 1822–1852. The aim of the work is indicated by its title. It has been much criticized, but the part on the general power and logarithms is meri-

[1] Gergonne, *Annales de math. p. et appl.*, T. XIV, 1823–4, p. 275.
[2] Same journal, T. xv, p. 110.

torious. These topics are treated in the second volume, of which the second edition, Berlin, 1829, lies before us. The first edition appeared in 1823. The edition of 1829 is the one usually quoted by contemporary writers. Ohm gave an exposition of his theory of logarithms as early as 1821, in a Latin thesis which had probably a very limited circulation.[1] We have never seen it quoted anywhere. Ohm introduces the terms *complete equation* and *incomplete equation.* An equation is *complete* when both sides of it have the same number of values representing exactly the same expressions; it is *incomplete* when one or several (but not all) of the many-valued expressions on the right and left are the same.[2] We have pointed out that Euler had occasion to consider these two types of equations in his paper of 1749 on logarithms. After having developed the eulerian theory of logarithms Ohm takes up the general power,[3] "allgemeine Potenz," a^x, where both a and x are complex numbers, namely $a = p + qi$, $x = \alpha + \beta i$. Assuming e^z as always single-valued, and letting $r = \sqrt{p^2 + q^2}$, $\log a = Lr + (\pm 2m\pi + \varphi)i$, he takes $a^x = e^{x \log a} = e^{\alpha . Lr - \beta(\pm 2m\pi + \phi)} \cdot \{\cos[\beta \cdot Lr + \alpha(\pm 2m\pi + \varphi)] + i \sin[\beta \cdot Lr + \alpha(\pm 2m\pi + \varphi)]\}$, where $m = 0, +1, +2, \cdots$ and L signifies the tabular logarithm. Thus the general power has an infinite number of values, but all are of the form $a + bi$. Ohm shows (1) that all of the infinite values are equal when x is an integer, (2) that there are n distinct values when x is a real, rational fraction s/n, (3) that some of the values are equal, though the number of distinct values is infinite, when x is real but irrational, (4) that the values are all distinct when x is imaginary. His idea of the irrational is partly explained by the statement that, if $x = s/n$ is irrational, then s and n are *unendlich grosse und nie angebbare Zahlen.* As illustrations of what his formula yields in special cases, Ohm shows that $i^i = (-i)^{-i} = e^{-(\pm 2d + \frac{1}{2})\pi}$, $d = 0, 1, \cdots$.

He inquires next, how the formulæ (A) $a^x \cdot a^y = a^{x+y}$, (B) $a^x \div a^y = a^{x-y}$ (C) $a^x \cdot b^x = (ab)^x$, (D) $a^x \div b^x = (a \div b)^x$, (E) $(a^x)^y = a^{xy}$ apply to the general exponent a^x, and finds that (A), (B) and (E) are incomplete equations, since the left members have " many, many more " values than the right members, although the right-hand values (infinite in number) are all found among the "infinite times infinite" values on the left; that (C) and (D) are complete equations for the general case.

Ohm next sets limitations upon the choice of values of *log a* and *log b*. When out of the infinite number of values of *log a*, some particular one, say α, is to be selected in the power $a^x = e^{x \log a}$, he indicates this by the notation $(a||\alpha)^x$. He writes similarly, $(b||\beta)^x$. With this understanding, we can write $x \log a + y \log a = (x + y) \log a$, and formula (A) becomes complete. The same is true of (B); while (C) and (D) remain complete equations under this restricted interpretation. It will be noticed that Ohm did not introduce the particular

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[1] *De innumerosis novis logarithmorum generibus. Dissertatio qua ad examen solemne in gymnasio regio Thorunensi die XIV. Aprilis MDCCCXXI publice habendum omnes literarum cultores invitat Dr. Martinus Ohm, nonnullarum societatum literarum sodalis.* Berolini, Typis Haynianis.

[2] M. Ohm, *System d. Math.* 2, Theil, Berlin, 1829, p. 386.

[3] Ohm, *op. cit.*, 2. Theil, 1829, p. 412.

value of a^x, which is now called the "principal value." Aside from that his treatment of the general power is mainly that of the present time, except, of course, in the explanation of the irrational.

Ohm proceeds to the general logarithm, and he makes the novel and significant statement that "with the concept of the general power is given the concept of the general logarithm $b?a$, if by this is meant every expression x, such that one has $a^x = b$ or $e^{x \log a} = b$."[1] Notice his notation for the general logarithm, $b?a$. As x has an infinite number of values for each value of log a, we have here the appearance of two independent and arbitrary constants, as in the researches of Stein and in those of Graves, Hamilton, and others, to be discussed later. He says that since b and a are taken completely general, and a^x has an infinity of values, $b?a$ is wholly undetermined, unless we state for what value of *log* a the power a^x is to be taken. He adopts the special notation $b?(a \| \alpha)$, which means the logarithm of b to the base a, when $\log a = \alpha$. Ohm shows that $b?(a \| \alpha) = (\log b)/\alpha$ is a complete equation. If a is positive, say e, then Ohm's logarithmic system reduces to the familiar eulerian form. "And if a is not positive, then the general logarithm, as well as each of its special cases is different from every one of the logarithms hitherto defined and discussed."

It must be granted that Ohm has surpassed all his predecessors in the generality and fullness of discussion of the expression a^x, and that he is the first writer to successfully base the general theory of logarithms (having a complex number as a base) fully and unreservedly upon the theory of the general power a^x. But the theory of the general power, as we have seen, is not developed independently of logarithms; in fact it uses the theory of logarithms of complex numbers to the base e. Thus, the eulerian logarithms have answered as a step-ladder leading to the theory of the general power; the theory of the general power, in turn, has led up to a more general theory of logarithms having a complex base.

The first original research on the logarithms of complex numbers published in England appeared in 1829, in the *Philosophical Transactions*, from the pen of John Graves, then a young man of 23. Graves was a class-fellow of William Rowan Hamilton in Dublin. Hamilton repeatedly expresses his indebtedness to Graves and states that reflecting on Graves's ideas on imaginaries led, finally, to his invention of quaternions. Graves became a noted jurist. His paper of 1829 bears the title: "An attempt to rectify the inaccuracy of some logarithmic formulæ." The author modifies $l1$ by certain rather startling extensions, yielding $l1 = (2m'\pi\sqrt{-1})/(1 + 2m\pi\sqrt{-1})$. This is done by taking the base e not in its *arithmetical* form, but in its more general form $e^{1+2m\pi\sqrt{-1}}$, so that x is defined as the general logarithm of y, not by the equation $e^x = y$, but by the equation $e^{(1+2m\pi\sqrt{-1})x} = y = e^{\log y + 2m'\pi\sqrt{-1}}$, where log y means the real or tabular logarithm of y. Letting $y = 1$ gives $l1$ as written above. Thus, Graves (like Ohms) claimed that in the general expression for the logarithm there are two arbitrary and independent integers, m and m', instead of simply one, as given by Euler.

[1] Ohm, *op. cit.*, 2. Theil, Berlin, 1829, p. 438.

In a paper of subsequent date, published in the same volume of the *Philosophical Transactions* (1829), the Rev. John Warren of Cambridge contributed a paper which we noticed when we were considering the geometric representation of algebraic quantities. Warren arrived at some of the algebraic results of Graves. In June, 1832, Vincent published, at Lille, results identical in effect with the principal formulæ of Graves.[1]

For lack of explicitness in Graves's writings exception was taken to his views by Augustus De Morgan in his treatise on the Calculus of Functions, sections 158 and 245, published in the *Encyclopaedia Metropolitana*.

In 1833 George Peacock made a report on the recent progress of certain branches of analysis in which he touches upon eulerian logarithms,[2] and argues that even in that theory negative numbers may have real logarithms. For, consider $-a^m$ as originating from $(-1)(+a)^m$, then $l(-a)^m = (2r+2mr'+1)\pi\sqrt{-1} + m \log a$. If we suppose $m = \frac{1}{2}$, $r = 0$, $r' = -1$, we have $l(-\sqrt{a}) = \frac{1}{2}\log a = l\sqrt{a}$. The reader will see that the eulerian theory is here violated by the assumption that in $2mr'$ it is possible to take r' odd. As regards the work of Graves, Peacock thought that there was a fundamental error in his generalization, because he makes a *periodic* quantity the base of his system. Graves sent a defence to the British Association in 1834, also to the *Philosophical Magazine*.[3] The outcome of the discussion was that Graves withdrew the statement contained in the title of his first paper, to the effect that he was attempting to "rectify the inaccuracy" of the eulerian theory, while De Morgan admitted that if Graves desired to extend the idea of a logarithm so as to include not only the logarithms of the arithmetical form of the base, but also those of a more general form, there was no error involved in the process. In fact, De Morgan suggested a still further extension.[4] Logarithmic systems like that of Graves, in which there are two arbitrary and independent integers were also worked out by Sir William R. Hamilton.[5]

Without being aware, apparently, that De Morgan and Graves had come to an understanding, D. F. Gregory of Trinity College, Cambridge, published an able article "On the Impossible Logarithms of Quantities,"[6] in which he inquires "which is the correct result," that of Graves or of his opponents. Though admitting the possibility of both, Gregory considers the one with an arithmetical base as more expedient. Gregory also points out that Vincent, Peacock and Graves, each erroneously thought he had established that in certain cases there is a logarithm in common for positive and negative numbers. Gregory died at the premature age of 31, yet in his short scientific career he did much toward establishing the foundations of algebra.

[1] *Report of the Fourth Meeting of the Brit. Assn.*, London, 1835, p. 524. We have not seen Vincent's publication of 1832.

[2] *Report of the Third Meeting of the British Ass'n*, London, 1834, p. 264.

[3] *Phil. Mag.*, Vol. VIII, 1836, p. 281.

[4] *Philos. Magazine*, Vol. IX, 1836, p. 252.

[5] "On Conjugate Functions or Algebraic Couples, etc.," *Transactions of the Royal Irish Acad.*, Vol. 17, Part II, 1835; *Lectures on Quaternions*, Dublin, 1853, preface, p. (12).

[6] *Cambridge Math'l Jour.*, Vol. I, 1837, p. 226.

Deep philosophic insight and logical power are evident in four papers *On the Foundation of Algebra,* published by Augustus De Morgan in 1842 and 1849. They were read before the Cambridge Philosophical Society in 1839, 1841, 1843 and 1844, respectively. From his first paper we quote the following:[1]

"If we define log x, or rather λx (reserving log x for the numerical logarithm of the length) to be any legitimate solution of $\epsilon^{\lambda x} = x$, it is plain that the logarithm of n inclined at an angle ν, (or of N) to the base b inclined at an angle β, (or B) is to be derived (avoiding ambiguity) from

$$(b\epsilon^{\beta\sqrt{-1}})^x = = n\epsilon^{\nu\sqrt{-1}},$$

or

$$\lambda_B N = = \frac{\log n + \nu\sqrt{-1}}{\log b + \beta\sqrt{-1}}.$$

This result is real when $\log n/\log b = \nu/\beta$; nor is it more surprising that an impossible quantity (hitherto so called) should have a possible logarithm, than that exponential operations not containing $\sqrt{-1}$, or not interchanging exponents of length and direction, should in certain cases enable us to pass from one line to another."[2]

It is worthy of note that in discussing the exponent $\sqrt{-1}$ on page 184, the author discloses, though somewhat vaguely, the notion of abstract groups and develops the cyclic group of order four. This paper was written by De Morgan in 1839, or some years before Cayley and William Rowan Hamilton came out with their researches on groups. It is one of the earliest, perhaps the earliest, of English references to group theory.

De Morgan shows that Graves's result is a special case of the above. In his second paper, De Morgan proceeds to the study of a^b. Reasoning geometrically as before and letting (r, ρ) be a line of r units inclined to the unit line at an angle ρ, he calls the *logometer* of (r, ρ) or $\lambda(r, \rho)$, the logarithm of that line, not only with respect to its length, but also the quantity of revolution by which it attained its present direction. From $e^{\pi\sqrt{-1}} = -1$ he gets $\pi = \lambda(-1)/(\sqrt{-1})$,

"a proposition which, not many years since, was one of the mysteries of analysis. It is now a very simple geometrical proposition: the first side means a line of π units laid down positively on the unit-line; the second side means the logometer of a negative unit turned back through a right angle. Now the logometer of a negative unit is a line of π units erected positively perpendicular to the unit-line: whence the identity of the two sides is manifest."[3]

These subjects are elaborated also in De Morgan's *Double Algebra,* London, 1849. He aimed to avoid the consideration of ambiguous values of symbols, "a thing for which there is no necessity."

"The more I think on this subject, the better satisfied do I feel, that the new algebra should have no symbols of double or multiple value whatsoever; that is, that the meaning of each elementary symbol should not be considered as complete, unless it expresses the amount of revolution from the unit line by which it is to be made to attain its direction, as well as that direction itself."[4]

He points out that the consequences of the full extension of a^b turn out to be capable of expression by the particular case in general use. "Whereas,[5] in the common system ϵ and $\epsilon^{\sqrt{-1}}$ are the logarithmic bases employed for ordinary and periodic magnitudes, we have, in the system above described, employed $e^{(m+n\sqrt{-1})^{-1}}$, and $e^{(\nu-\mu\sqrt{-1})^{-1}}$."

[1] *Trans. of the Cambridge Philos. Society,* Vol. 7, 1842, p. 186.

[2] The double sign of equality is used by De Morgan to indicate "that every symbol shall express not merely the length and direction of a line, but also, the quantity of revolution by which a line, setting out from the unit line, is supposed to attain that direction."

[3] *Trans. Cambr. Phil. Soc.,* Vol. 7, 1842, p. 294.

[4] De Morgan, *Trans. Cambr. Phil. Soc.,* Vol. 7, p. 296.

[5] *Trans. Cambr. Phil. Soc.,* Vol. VIII, Pt. II, p. 3.

An independent discussion of the generalized power $A^B = C$ is given by G. M. Pagani,[1] professor at the Catholic University of Löwen. He takes $A = \mu e^{\theta i}$, where the argument θ is called the *déterminatif* of A. The consideration of the *déterminatif* of a quantity is the only means, says the author, of avoiding paradoxes in algebraic analysis. He uses the notation $(a)^{m/n}$ to indicate all the n values of $a^{m/n}$. Assuming A and C as given, he calculates B. Letting $A = \mu e^{i\theta} = e^{l\mu + i\theta}$, $B = \alpha + i\beta$, he gets $C = e^{(l\mu + i\theta)(\alpha + i\beta)} = re^{i\phi}$ and finally $\alpha = (l\mu + \theta\phi) \div (l^2\mu + \theta^2)$, $\beta = (\varphi l\mu - \theta l\gamma) \div (l^2\mu + \theta^2)$. Here l means the tabular logarithm to the base e; θ and ϕ are each infinitely many-valued. Pagani points out that, if A is real and positive, so that $\theta = 2k\pi$, and if $\varphi = 2k\pi \cdot lr \div l\mu$, then the imaginary number $re^{i\phi}$ has a real logarithm. This conclusion is evidently erroneous. He says that his formulæ are *plus générales et par conséquent plus exactes* than those of Euler, since Euler's are obtained by letting $k = 0$. Pagani's paper was read before the Academy at Bruxelles on October 7, 1837. After the lecture Quetelet pointed out that J. S. Cerquero,[1] director of the observatory at San-Fernando, near Cadix, had reached similar results.[2] Later, Pagani prepared a note[3] on the limit of $(1+(x/n)^n$, x being complex, whereupon he was informed of the work of Graves in England on A^B. We shall see that the general power was treated also by Cauchy in 1847.[4]

Pagani's error, referred to above, arose from the tacit assumption that $(A^x)^y = A^{xy}$ is a complete equation. Ohm had shown that it is incomplete. A failure to pay attention to this matter leads to curious paradoxes. Th. Clausen of Altona gave such a paradox in 1827.[5] It was quoted by Peacock in his report.[6] When n is an integer, $e^{2n\pi i} = 1$, $e^{1+2n\pi i} = e$, hence also $e^{(1+2n\pi i)^2} = e = e^{1+4n\pi i - 4n^2\pi^2}$. Since $e^{1+4n\pi i} = e$, it would follow that $e^{-4n^2\pi^2} = 1$, which is absurd. In more condensed form the paradox is presented by E. Catalan[7] thus: $e^{2m\pi i} = e^{2n\pi i}$, where m and n are any two distinct integers. Raising both sides to the power $i/2$, we have the absurdity, $e^{-m\pi} = e^{-n\pi}$.

The reader will notice that $(e^{2m\pi i})^{i/2} = e^{-m\pi}$ is an incomplete equation; all values of the right member are values of the left member, but not *vice versa*. Let $((a))^x$ signify all the values of the general power. Since $((a))^x = e^{x \log a + 2p\pi i}$ is a *complete* equation, where $\log a = la + 2q\pi i$, $la =$ tabular logarithm, $\pm p = 0, 1, 2, \cdots$ and $\pm q = 0, 1, 2, \cdots$, we obtain the following complete equations $((\,((e))^{2m\pi i}))^{i/2} = ((e^{2m\pi i \log e + 2p\pi i}))^{i/2} = e^{-m\pi - p\pi - 2mq\pi^2 i}$. When $p = q = 0$, we have the particular value e^{-mx}. Similar results are obtained for $(e^{2n\pi i})^{i/2}$. It follows that $(e^{2m\pi i})^{i/2} = (e^{2n\pi i})^{i/2}$ is an incomplete equation, in which the

[1] *N. mémoires de l'academie royale des sciences et belles-lettres de Bruxelles*, T. XI, Bruxelles, 1838, pp. 1–11.

[2] We have been unable to find any other reference to Cerquero's research.

[3] *Bulletins de l'académie r. d. scien. et b. l. de Bruxelles*, T. 6 (*1re partie*), *Bruxelles*, 1839, p. 256.

[4] Cauchy, *Exercices d'analyse et de phys. math.*, T. IV, 1847, p. 255.

[5] *Crelle's Journal*, Vol. 2, Berlin, 1827, pp. 286–287.

[6] *British Ass'n Report*, London, 1834, p. 347.

[7] *Nouv. Ann. de Math.*, 2e S., Vol. 8, 1869, p. 456. See also Vallès in the same journal, 2e S., Vol. 9, 1870, p. 20.

particular value e^{-mx} on the left is not equal to the particular value e^{-nx} on the right. This last equation becomes complete under the restrictions $m + p = n+p'$ and $mq = nq'$, where the integers p' and q' are used in the complete value of the right member in the same way as p and q were used on the left. De Morgan draws upon the notion of direction of lines to explain paradoxes of this character.[1] He reminds us that such fallacies have been seriously proposed as arguments against the use of imaginary quantities.

The generalizations of logarithms leading to systems having periodic or complex bases gave rise to formulæ of such complexity that the generalized system failed of adoption. The machinery was too intricate for general use. The theory of the general power a^b, a and b being complex numbers, was established satisfactorily with the aid of the eulerian logarithms to the base $e = 2.718$. No real advantage grew out of the new logarithmic systems. If in $a^x = c$, the complex numbers a and c are given and x is to be found, the eulerian logarithmic formulæ are sufficient for the solution. Thus it happened that the general logarithmic systems of Ohm, Graves, Vincent, Warren, De Morgan, Gregory, Hamilton and Pagani failed of recognition as useful mathematical inventions.

However, one result of the study of the general power and of logarithms having a number $a + ib$ as a base was of importance; this study led to the conclusion that no possible combination of real and ordinary complex numbers gives rise to new forms of numbers. De Morgan frankly admits that he had expected some "new *imaginary* or *impossible* quantities."[2] But the processes of addition and subtraction, multiplication and division, involution and its two inverses, evolution and logarithmation, constitute a closed group of operations; they complete the cycle of operations in the algebra of ordinary complex numbers. In the seventeenth century the logarithmation of negative and complex numbers was not recognized; hence the cycle of operations was not a closed one and the algebra of that time was not complete.

Similar researches were carried on for quaternions. We have seen that William Rowan Hamilton was in close touch with the work of J. T. Graves and De Morgan. He was familiar with the investigations of Ohm. Like Cauchy and others, Hamilton recognized the great inconvenience arising from the multiplicity of logarithmic values. ". . ., but it has been my object, in the present theory, to preclude, so far as I could, that indeterminateness by *definition*."[3]

A difficulty in Hamilton's development of general quaternion theory—his algebra of space—is described by him as follows:

> In the present theory of diplanar quaternions, we cannot expect to find that the sum of the logarithms of any two proposed factors shall be generally equal to the logarithm of the product; but for the simpler and easier case of complanar quaternions, that algebraic property may be considered to exist, with due modification for multiplicity of value.[4]

More recently this difficulty has received the attention of Alexander Mac-

[1] De Morgan, *Double Algebra*, p. 136.
[2] *Trans. Cambr. Phil. Soc.*, Vol. VIII, Pt. II, 1844, pp. 3, 4.
[3] W. R. Hamilton, *Lectures on Quaternions*, Dublin, 1853, pp. (15), 556.
[4] W. R. Hamilton, *Elements of Quaternions*, London, 1866, Section 11, p. 386.

farlane. He points out that in the algebra of the plane there is no need of specifying the axis of rotation for a circular angle, such as e^{ib}. It is different in space. When angles on a sphere are considered, the axis must be inserted, and also $\pi/2$ for the indefinite i.[1] If by α^a we mean a radians round the axis α, α^1 means 1 radian round α. Here $\log \alpha^1 = \alpha^{\pi/2}$ and $\log \alpha^a = a \log \alpha^1 = a\alpha^{\pi/2}$. Whence $\alpha^a = e^{a \log_e \alpha^1} = e^{a\alpha^{\pi/2}}$. Now Hamilton adopted the principle which, in Macfarlane's notation is $\alpha^{\pi/2}\beta^{\pi/2} = -\cos\alpha\beta + \sin\alpha\beta[\alpha\beta]^{\pi/2}$, where $[\alpha\beta]$ signifies the unit which is normal to α and β. Macfarlane, on the other hand, adopts the principle $\alpha^{\pi/2}\beta^{\pi/2} = -\cos\alpha\beta - \sin\alpha\beta\,[\alpha\beta]^{\pi/2}$. By this modification Macfarlane is able to establish for his space-algebra the simple theorem which Hamilton could not derive from his assumption, $\log(e^{a\alpha^{\pi/2}} \cdot e^{b\beta^{\pi/2}}) = a\alpha^{\pi/2} + b\beta^{\pi/2}$.

A SIMPLE FORMULA FOR THE ANGLE BETWEEN TWO PLANES.

By EDWARD V. HUNTINGTON, Harvard University.

Let us consider the familiar problem of finding the area of a surface by the methods of the integral calculus.

Let S be any closed portion of a surface whose equation is $z = f(x, y)$, and let ΔS be an "element" of the surface at any point P. Then, as is well known,

$$\Delta S = \frac{\Delta S'}{\cos\gamma},$$

where $\Delta S'$ is the projection of ΔS on the xy-plane, and γ is the angle between the xy-plane and the tangent plane to the surface at P. The area of S is therefore given by

$$S = \int\int \frac{dS'}{\cos\gamma},$$

where the integration is to be extended over the whole of the plane area S' which is the projection of S on the xy-plane.

The problem of finding the area S thus reduces to a simple problem in double integration, *provided the value of γ at every point of the surface is known.*

In the ordinary treatment of the subject in the text-books, the derivation of the formula for γ requires a considerable knowledge of analytical geometry of three dimensions, including the equation of a plane, the equations of a straight line normal to a plane, the method of finding the direction cosines of a line when its equations are given, the equation of a tangent plane, etc.

The object of this note is *to show how the angle γ may be found directly*, by very elementary methods. The formula obtained, in the easily-remembered form here given, and the methods of derivation are so simple that it is hardly possible that they are new; and yet I do not find them in any of the current text-books.

[1] A. Macfarlane, "A vector-analysis as generalized algebra," *International Congress of Mathematicians*, Cambridge, August, 1912; "Account of researches in the algebra of physics," *Journal of the Washington Acad. of Sci.*, Vol. II, Nos. 14, 15, 16, 1912.

The Formula for the Angle γ.

Let P be any point of the surface, and let PA and PB (see either figure) be the lines in which the tangent plane at P intersects the vertical planes drawn through P parallel to the planes xz and yz respectively.

The angles φ_x and φ_y which PA and PB make with the horizontal can be readily determined from the fact that tan φ_x is the partial derivative of z with respect to x, and tan φ_y is the partial derivative of z with respect to y; thus,

$$\tan \varphi_x = f_x'(x, y) = D_x z, \quad \tan \varphi_y = f_y'(x, y) = D_y z.$$

We may therefore suppose that tan φ_x and tan φ_y are known for every point P of the surface.

Then if γ is the angle which the tangent plane at P makes with the base plane, the value of tan γ may be at once computed by the easily remembered formula

$$\tan^2 \gamma = \tan^2 \varphi_x + \tan^2 \varphi_y,$$

whence, of course, $\sec \gamma = \sqrt{1 + \tan^2 \gamma} = \sqrt{1 + (D_x z)^2 + (D_y z)^2}$.

The proof of the formula may be obtained by either of two methods, as follows.

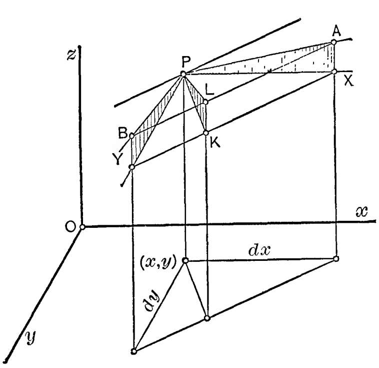

Fig. 1.

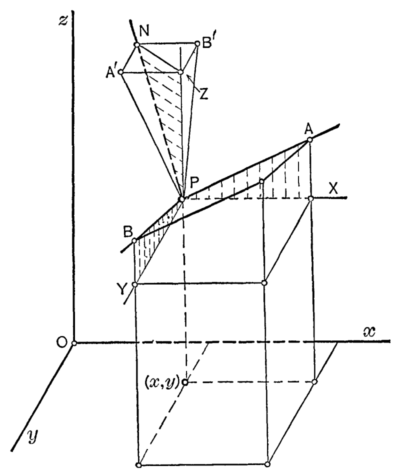

Fig. 2.

First Proof.—Referring to figure 1, we may suppose that the values of dx and dy have been so chosen that the perpendiculars AX and BY dropped from A and B on the horizontal plane through P shall be equal, say $AX = BY = h$. Then

$$\tan \varphi_x = \frac{h}{dx} \quad \text{and} \quad \tan \varphi_y = \frac{h}{dy}.$$

Now since $AX = BY$, the lines AB and XY will be parallel to the line of

intersection of the tangent plane with the horizontal plane. If, therefore, we draw PK perpendicular to XY and PL perpendicular to AB, then PK and PL will be perpendicular to the edge of the dihedral angle γ, so that the angle KPL will be the correct measure of that dihedral angle. Hence

$$\tan \gamma = \frac{h}{PK}.$$

But, by considering the right triangle PXY, we find at once that

$$\frac{1}{PK^2} = \frac{1}{dx^2} + \frac{1}{dy^2}.$$

Therefore

$$\tan^2 \gamma = \frac{h^2}{PK^2} = \frac{h^2}{dx^2} + \frac{h^2}{dy^2} = \tan^2 \varphi_x + \tan^2 \varphi_y,$$

which was to be proved.

Second Proof.—Through the given point P, in figure 2, draw PX, PY, PZ parallel to the coördinate axes, and let PN be the normal to the surface at P, so that $\angle\ ZPN = \gamma$.

In the plane PXZ draw PA' perpendicular to PA, so that $\angle\ ZPA' = \angle\ XPA = \varphi_x$, and in the plane PYZ draw PB' perpendicular to PB, so that $\angle\ ZPB' = \angle\ YPB = \varphi_y$; and let N, A', B', be the points of the lines PN, PA', PB' which lie in a horizontal plane through Z. Then

$$\tan \gamma = \frac{ZN}{PZ}, \qquad \tan \varphi_x = \frac{ZA'}{PZ}, \qquad \tan \varphi_y = \frac{ZB'}{PZ}.$$

If now we can show that the figure $ZA'NB'$ is a rectangle, we shall have

$$ZN^2 = ZA'^2 + ZB'^2,$$

and hence $\tan^2 \gamma = \tan^2 \varphi_x + \tan^2 \varphi_y$, as desired.

We prove first that NA' is parallel to ZB'.

From the figure, PN is perpendicular to PA, PA' is perpendicular to PA, and PY is perpendicular to PA. Hence PA' lies in the plane PNY. But this plane cuts the horizontal plane through Z in a line NA' which is parallel to PY and hence parallel to ZB'.

Similarly we can prove that NB' is parallel to ZA'.

Hence the figure $ZA'NB'$ is at least a parallelogram. But the angle between ZA' and ZB' is known to be a right angle. Hence the figure is a rectangle, and the proof is complete.

It will be noticed that the distinction between the two methods of proof consists mainly in the fact that the required angle γ is regarded as the *dihedral angle* between the planes in the first proof, and as the *angle between the normals* to those planes in the second proof. These angles are of course equal.

A modification of the first proof may be obtained by considering *two* planes perpendicular to the edge of the dihedral, one containing AX and the other containing AY, instead of the single plane containing P.

A DIRECT DEFINITION OF LOGARITHMIC DERIVATIVE.*

By E. R. HEDRICK, University of Missouri.

1. The Usual Definitions. The logarithmic derivative of a function $y = f(x)$ is usually defined either as the derivative of the Napierian logarithm of y:

$$\frac{d \log_e y}{dx} = \frac{d \log_e f(x)}{dx}; \tag{1}$$

or by means of the resulting formula:

$$\frac{dy}{dx} \div y = f'(x) \div f(x). \tag{2}$$

These definitions suffer from the fact that the logarithm is defined only when $f(x)$ is positive, though (2) may be used independently of (1) when $f(x)$ is negative.

2. A Direct Definition. The importance of logarithmic differentiation in elementary work arises from its meaning as the *relative rate of increase* of y with respect to x. From this standpoint, the intervention of logarithms in the definition is wholly accidental and extraneous.

In order to define directly the relative rate of increase, we may divide the actual increase Δy by the increase in x, Δx, divide this ratio $\Delta y/\Delta x$ by some average value of $f(x)$ in the range Δx, and take the limit of this quotient as Δx approaches zero. Denoting the relative rate of increase by r_r, we may write

$$r_r = \lim_{\Delta x = 0} \left\{ \frac{\Delta y}{\Delta x} \div [\text{Average Value of } f(x)] \right\}. \tag{3}$$

The average value of $f(x)$ to be used may be selected in a variety of ways; that which seems most appropriate is the usual expression:

$$[\text{Average Value of } f(x)] = \frac{1}{\Delta x} \int_{x=a}^{x=a+\Delta x} f(x)dx,$$

where $x = a$ is the value of x at which the relative rate is to be found. Then (3) becomes

$$r_r = \lim_{\Delta x=0} \left\{ \frac{\Delta y}{\Delta x} \div \left[\frac{1}{\Delta x} \int_{x=a}^{x=a+\Delta x} f(x)dx \right] \right\},$$

whence, simplifying and replacing Δy by $f(a + \Delta x) - f(a)$, we obtain

$$r_r = \lim_{\Delta x=0} \left[\frac{f(a + \Delta x) - f(a)}{\int_a^{a+\Delta x} f(x)dx} \right]. \tag{4}$$

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* Read before the Southwestern Section of the American Mathematical Society, November, 1912.

Let $\phi(x)$ be any indefinite integral of $f(x)$; then

$$\int f(x)dx = \phi(x), \qquad \int_a^{a+\Delta x} f(x)dx = \phi(a + \Delta x) - \phi(a),$$

and (4) becomes

$$r_r = \lim_{\Delta x = 0} \left[\frac{f(a + \Delta x) - f(a)}{\phi(a + \Delta x) - \phi(a)} \right], \tag{5}$$

which we shall adopt as *the definition of the relative rate of increase of $f(x)$, or the logarithmic derivative of $f(x)$, with respect to x.*

3. Identification with the Usual Definition. At least when $f(x)$ is positive, the definition (5) coincides with the usual definition (1); and (5) coincides with the definition (2) whenever (2) has a meaning.

For, since

$$\frac{f(a + \Delta x) - f(a)}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{\phi(a + \Delta x) - \phi(a)} \cdot \frac{\phi(a + \Delta x) - \phi(a)}{\Delta x} \tag{6}$$

and since the limit of $[\phi(a + \Delta x) - \phi(a)]/\Delta x$ surely exists and is equal to $f(a)$, if $f(x)$ is continuous at $x = a$, we have

$$f'(a) = r_r \cdot f(a) \tag{7}$$

provided $f'(a)$ exists.

Conversely, if r_r, as defined by (5), exists, and if $f(x)$ is continuous at $x = a$, it follows from (6) that $f'(a)$ exists.

4. The Law of the Mean. If $f(x)$ is continuous, and if r_r exists, the derivative $f'(x)$ exists; moreover $\phi'(x)$ exists and is equal to $f(x)$; hence, by the generalized law of the mean,*

$$\frac{f(a + \Delta x) - f(a)}{\phi(a + \Delta x) - \phi(a)} = \frac{f'(c)}{\phi'(c)} = \frac{f'(c)}{f(c)}, \tag{8}$$

where c lies between a and $a + \Delta x$.

The expression on the left, $[f(a + \Delta x) - f(a)] \div [\phi(a + \Delta x) - \phi(a)]$, may be called the average relative rate over the interval Δx; with this notation, the equation (8) may be stated as follows: *The average relative rate over the interval Δx is precisely equal to the instantaneous relative rate at some point in that interval.*

This statement, which is the analogon to the law of the mean for ordinary derivatives, may be used to prove directly several important theorems.

Thus, *if the logarithmic derivative is zero identically, it follows from* (8) *that $f(x)$ is a constant.*

Again, suppose that the logarithmic derivatives of two functions $f(x)$ and $F(x)$ are identically equal; then

$$0 = \frac{f'(x)}{f(x)} - \frac{F'(x)}{F(x)} = \frac{f'(x)F(x) - F'(x)f(x)}{[F(x)]^2} \div \frac{f(x)}{F(x)}.$$

* See, for example, Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 8; it is vital that c is the same in numerator and denominator.

The extreme right-hand member is, however, the logarithmic derivative of $f(x)/F(x)$; hence by the preceding theorem, $f(x)/F(x)$ is constant, or $f(x) = k \cdot F(x)$: *If two functions have the same logarithmic derivative, their quotient is a constant*; or *If the logarithmic derivative of a function is given, that function is determined except for a constant factor.*

These are the fundamental theorems concerning logarithmic derivatives; they are precisely analogous to the corresponding theorems for ordinary derivatives.

5. Discontinuity. It has been shown above that $f'(x)$ exists whenever the relative rate r_r defined by (5) exists, provided only that $f(x)$ is continuous. It is, however, possible that $f'(x)$ and therefore also r_r is discontinuous.

It might appear that (8) precludes such a possibility, for the left side approaches r_r, and c approaches a as Δx approaches zero. But c does not necessarily take on all values; hence the approach of c to a may be only through a set of special values of x.

That the possibility just mentioned actually does occur is manifested by the example

$$f(x) = 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x} + 1, \qquad x \neq 0; \qquad f(0) = 1;$$

which has an indefinite integral:

$$\phi(x) = x^4 \sin\frac{1}{x} + x, \qquad x \neq 0; \qquad \phi(0) = 0.$$

Its derivative is

$$f'(x) = 12x^2 \sin\frac{1}{x} - 5x \cos\frac{1}{x} + \sin\frac{1}{x}, \qquad x \neq 0; \qquad f'(0) = 0;$$

which is discontinuous. Its logarithmic derivative may be derived either by (2) or by (5), and its value when $x = 0$ is zero.

6. Conclusion. The facts and illustrations given above parallel completely the fundamental theorems and illustrations usually given for ordinary derivatives. In deducing them, although the name *logarithmic* derivative has been used, the notion of logarithms has not been employed, and it has been shown that the ideas themselves are independent of the concept of logarithms.

TWO GEOMETRICAL APPLICATIONS OF THE METHOD OF LEAST SQUARES.

By J. L. COOLIDGE, Harvard University.

The two problems discussed in the present paper are taken from a recent work by Vahlen entitled " Konstruktionen und Approximationen."* It seems to me that they are sufficiently interesting and important to merit special attention.

* Leipzig, 1911, pages 125, 126.

Problem 1. *n lines are drawn in a plane which should be concurrent, but are not. If the lines be equally trustworthy, what point should be taken for their expected point of concurrence?*

The solution given (without proof) for this problem is that of Bertot.† It seems rather artificial, and it occurred to me that a more direct solution would have a certain interest. We wish to find such a point that the sum of the squares of its distances from n given lines shall be a minimum. The lines being given in normal form, we have

$$\sum_{i=1}^{i=n} (x \cos \alpha_i + y \sin \alpha_i - p_i)^2 = \text{Minimum}.$$

Equating to zero the partial derivatives as to x and y,

$$\begin{aligned}\sum_{i=1}^{i=n} x \cos^2 \alpha_i + \sum_{i=1}^{i=n} y \cos \alpha_i \sin \alpha_i &= \sum_{i=1}^{i=n} p_i \cos \alpha_i,\\ \sum_{i=1}^{i=n} x \cos \alpha_i \sin \alpha_i + \sum_{i=1}^{i=n} y \sin^2 \alpha_i &= \sum_{i=1}^{i=n} p_i \sin \alpha_i.\end{aligned} \tag{1}$$

Let us suppose that a unit circle has been drawn about the origin as center; the points (1, 0) and (0, 1) shall be called A and B respectively. Circles are drawn on OA and OB as diameters. A perpendicular from the origin on the ith line shall meet that line in P_i, while it meets the OA and OB circles in Q_i and R_i respectively. See Fig. 1. We have for the coördinates of these various points

$$P_i = (p_i \cos \alpha_i,\ p_i \sin \alpha_i), \quad Q_i = (\cos^2 \alpha_i,\ \cos \alpha_i \sin \alpha_i),$$
$$R_i = (\cos \alpha_i \sin \alpha_i,\ \sin^2 \alpha_i).$$

The centre of gravity P of the n points P_i is easily constructed, as are Q, R the centres of gravity of Q_i and R_i. Let their coördinates be

$$P = (\bar{x}, \bar{y}), \quad Q = (x_1, y_1), \quad R = (x_2, y_2), \quad y_1 = x_2.$$

Equations (1) will take the form

$$x_1 x + y_1 y = \bar{x}, \quad x_2 x + y_2 y = \bar{y},$$

$$x = \frac{\begin{vmatrix} \bar{x} & x_2 \\ \bar{y} & y_2 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}} = \frac{\Delta OPR}{\Delta OQR} = \frac{\text{Distance } P - OR}{\text{Distance } Q - OR},$$

$$y = \frac{\begin{vmatrix} x_1 & \bar{x} \\ y_1 & \bar{y} \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}} = \frac{\Delta OQP}{\Delta OQR} = \frac{\text{Distance } P - OQ}{\text{Distance } R - OQ}.$$

† "Solution géométrique du problème de la détermination du lieu le plus probable du navire," *Comptes Rendus*, Vol. LXXXII, 1876, p. 682.

Since we know the unit length already, the values x and y are easily found, and the required point determined.

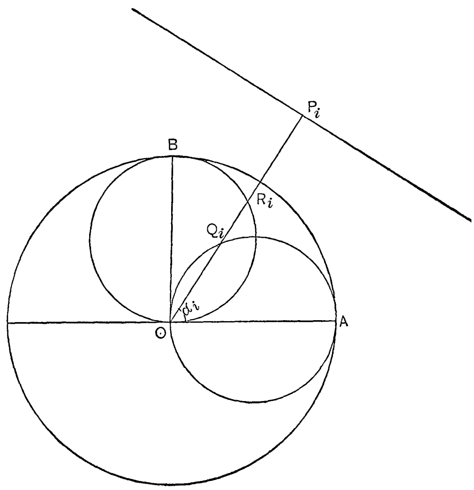


FIG. 1.

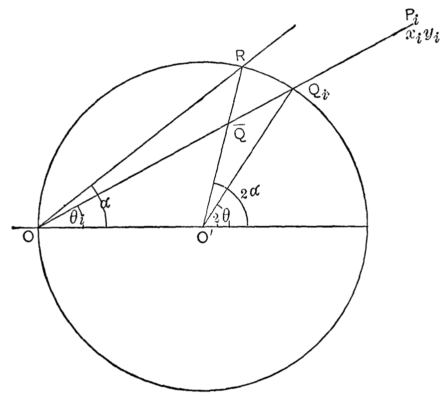


FIG. 2.

Problem 2. *n points are given in the plane which should be collinear, but are not. If the points be equally trustworthy, what line should be taken as their supposed common line?*

Vahlen's remark on this problem is the following:* "Für die konstruktive Ausgleichung einer Geraden aus mehr als zwei Punkten . . . scheinen dem Bertotschen entsprechende einfache Verfahren noch nicht gefunden zu sein. Sie wurden, wenn auch theoretisch von Interesse, doch praktisch schon zu kompliziert sein, um wirklich angewendet werden zu können." Without attempting to settle the obscure question of how complicated a construction may be without losing its practical importance, let us show that this problem is really easier of solution than the other. The given points shall be $(x_1, y_1) \cdots (x_n, y_n)$. We seek a line, the sum of the squares of whose distances from these shall be a minimum. If this line be given in normal form, we have

$$\sum_{i=1}^{i=n} (x_i \cos\alpha + y_i \sin\alpha - p)^2 = \text{Minimum}.$$

Equating to zero the partial derivative as to p, and dividing by n,

$$\frac{\sum_{i=1}^{i=n} x_i}{n} \cos\alpha + \frac{\sum_{i=1}^{i=n} y_i}{n} \sin\alpha - p = 0.$$

This shows that the line must at any rate pass through the center of gravity of all the points. Let us imagine that the origin has been moved to that point.

* Loc. cit., p. 126.

$$\sum_{i=1}^{i=n} (x_i \cos \alpha + y_i \sin \alpha)^2 = \text{Minimum}.$$

Equating to zero the derivative as to α,

$$\sum_{i=1}^{i=n} (y_i^2 - x_i^2) \sin \alpha \cos \alpha = \sum_{i=1}^{i=n} x_i y_i (\sin^2 \alpha - \cos^2 \alpha).$$

Let

$$\frac{y_i}{x_i} = \theta_i,$$

$$\tan 2\alpha = \frac{\sum_{i=1}^{i=n} \sin 2\theta_i}{\sum_{i=1}^{i=n} \cos 2\theta_i}.$$

The origin shall be O, the point (1, 0) shall be O'. We construct a unit circle about O' as center. Let OP_i meet this circle again in Q_i. The coördinates of Q_i in the system *where O' is origin* will be $(\cos 2\theta_i, \sin 2\theta_i)$. Let $\bar{Q}$ be the center of gravity of the points Q_i. The line $O'\bar{Q}$ shall meet the circle in R, that is to say, R is the intersection which lies on the same side of the diameter OO' as does $\bar{Q}$. Then OR is the required line. See Fig. 2.

COMPUTATION FORMULA FOR THE PROBABILITY OF AN EVENT HAPPENING AT LEAST C TIMES IN N TRIALS.

By E. C. MOLINA, New York City.

Many problems in biometry, radioactivity, etc., require for their quantitative solution a convenient formula for computing the probability of an event happening at least c times in n trials; the probability p of its happening in one trial being known. Several relatively simple formulas are given by Poisson in the third chapter of his *Recherches sur la Probabilité des Jugements*, availing himself of a method developed by Laplace in the *Théorie Analytique des Probabilités*. But these formulas explicitly exclude the case where p is very small and $c = np + r$, r not being small compared with np. For this range of values a problem connected with my engineering work forced me early in 1908 to develop the formulas* (5) and (6) given below.

Let $P =$ the required probability, $\alpha = np$, $s = (n - c)/(c + 1)$,

$$F(x) = \frac{2c(1 - x)x^2 + (n - s)sx^4}{[c - (n - s)x]^4}$$

* These formulas have been used for the construction of curves which are in constant use in the Engineering Department of the American Telephone and Telegraph Company. They are published in the hope that others will find them helpful.

and take as a starting point Poisson's exact equation

$$P = \frac{\int_\sigma^\infty Ydy}{\int_0^\infty Ydy}, \qquad \text{where} \qquad Y = \frac{y^{n-c}}{(1+y)^{n+1}} \qquad \text{and} \qquad \sigma = \frac{1-p}{p}.$$

Transforming the numerator by the change of variable $y = (1/x) - 1$ and substituting for the integral in the denominator its well-known value, gives

$$(1) \qquad P = \frac{n!}{(c-1)!\,(n-c)!}\int_0^p x^{c-1}(1-x)^{n-c}dx,$$

or, as

$$c = np + r > (n-s)p,$$

we may write

$$P = \frac{n!}{(c-1)!\,(n-c)!}\int_0^p \frac{\partial}{\partial x}\{x^c(1-x)^{n-c-s}\}\,\frac{(1-x)^{s+1}}{[c-(n-s)x]}\,dx.$$

By partial integration

$$(2) \qquad P = \frac{n!}{(c-1)!\,(n-c)!}\left\{\frac{p^c(1-p)^{n-c+1}}{[c-(n-s)p]} - (n-s)s\int_0^p \frac{x^{c+1}(1-x)^{n-c}}{[c-(n-s)x]^2}dx\right\}.$$

Also, by partial integration

$$(3) \qquad \begin{aligned}\int_0^p \frac{x^{c+1}(1-x)^{n-c}}{[c-(n-s)x]^2}dx &= \int_0^p \frac{\partial}{\partial x}\{x^c(1-x)^{n-c-s}\}\,\frac{x^2(1-x)^{s+1}}{[c-(n-s)x]^3}\,dx \\ &= \frac{p^{c+2}(1-p)^{n-c+1}}{[c-(n-s)p]^3} - \int_0^p x^{c-1}(1-x)^{n-c}F(x)dx.\end{aligned}$$

Now, since $F(x)$ remains finite, does not change sign and is a maximum for $x = p$ as x varies from 0 to p, provided $p < 2/3$,

$$(4) \qquad \int_0^p x^{c-1}(1-x)^{n-c}F(x)dx = \lambda F(p)\int_0^p x^{c-1}(1-x)^{n-c}dx, \qquad 0 < \lambda < 1.$$

By (2), (3), (4) and the expressions for s and $F(p)$ we obtain

$$(5) \qquad \begin{aligned}P = \frac{n!}{c!\,(n-c)!}p^c(1-p)^{n-c}&\left\{\frac{(1-p)(c+1)}{c+1-(n+1)p}\right\} \\ &\times\left\{1 - \frac{(\alpha+p)(\alpha-cp)}{c(c+1-\alpha-p)^2}\right\}\left\{\frac{1}{1-\lambda\varphi}\right\}\end{aligned}$$

where

$$\varphi = \frac{(\alpha+p)(\alpha-cp)\{2(c+1)^2(1-p)+(\alpha+p)(\alpha-cp)\}}{c^2(c+1-\alpha-p)^4} < \frac{\alpha^2\{2(c+1)^2+\alpha^2\}}{c^2(c+1-\alpha-p)^4}.$$

For p not greater than 1/30 and P not greater than .01 I found that the values of c and α $(= np)$ are such that φ is negligible. For this range of values, therefore, the factor $1/(1 - \lambda\varphi)$ in the right-hand member of (5) was ignored and a convenient formula for computing P thereby obtained. After plotting several curves it became evident that for a given value of P the corresponding value of c did not change appreciably with changes in the values of p and n, provided the product α $(= np)$ was kept constant. The following formulas were therefore used.

In (5) let n and p approach the limits ∞ and 0 respectively while α remains constant. Then in the limit

$$
(6) \qquad
\begin{aligned}
P &= \left\{\frac{1}{1-\lambda\psi}\right\}\left\{\frac{\alpha^c\epsilon^{-\alpha}}{c!}\right\}\left\{\frac{c+1}{c+1-\alpha}\right\}\left\{1-\frac{\alpha^2}{c(c+1-\alpha)^2}\right\}, \\
\psi &= \frac{\alpha^2\{2(c+1)^2+\alpha^2\}}{c^2(c+1-\alpha)^4}.
\end{aligned}
$$

Let P_0, P_1 be the values of P obtained from (6) by taking λ equal to 0 and 1 respectively. Then

$$
(7) \qquad P_0 < P < P_1.
$$

Therefore $(P_1 + P_0)/2$ may be taken as an approximate value of P provided $(P_1 - P_0)/2$ is small compared with P_0. This will be the case if P is not greater than .01 and α/c small. As α/c increases the discrepancy between P and $(P_1 + P_0)/2$ increases, but not indefinitely. This may be proved as follows.

For a given value of P we know (Bernoulli-Laplace Theorem) that α/c approaches the limit 1 as α and c approach ∞. Also, for α and c very large, one of Poisson's formulas is a very close approximation and reduces to

$$
(8) \qquad P = \frac{2}{\sqrt{\pi}}\int_\kappa^\infty \epsilon^{-t^2}dt
$$

where κ, the lower limit of the integral, is given by the equation

$$
\begin{aligned}
\frac{\kappa^2+(c-1-\alpha)}{(c-1)} &= -\log\left\{\frac{\alpha}{c-1}\right\} = -\log\left\{1-\frac{c-1-\alpha}{c-1}\right\} \\
&= \frac{(c-1-\alpha)}{(c-1)}+\frac{(c-1-\alpha)^2}{2(c-1)^2}+\frac{(c-1-\alpha)^3}{3(c-1)^3}+\cdots,
\end{aligned}
$$

or, for c very large and $\alpha/c \doteq 1$

$$
(9) \qquad \kappa^2 = \frac{(c-1-\alpha)^2}{2(c-1)} = \frac{(c+1-\alpha)^2}{2\alpha},
$$

but for $\alpha/c \doteq 1$

$$
(10) \qquad \psi = \frac{\alpha^2\{2(c+1)^2+\alpha^2\}}{c^2(c+1-\alpha)^4} = \frac{3\alpha^2}{(c+1-\alpha)^4} = \frac{3}{(2\kappa^2)^2}.
$$

Therefore, by (10) and any of the well-known tables giving the values of P and κ which satisfy equation (8), we obtain the following figures showing the limits (as $c \doteq \infty$ and $\alpha/c \doteq 1$) of the possible error incurred by taking $P = (P_1 + P_0)/2$.

P	κ^2	ψ	$(P_1 - P_0)/2P_0$
.01	2.72	.101	.056
.001	4.80	.032	.017
.0001	6.92	.016	.008

The short table which follows gives some of the pairs of values of c and α satisfying the three equations $P = .01$, $P = .001$ and $P = .0001$ for the limiting case of $p \doteq 0$, $n \doteq \infty$.

	$P = .0001$		$P = .001$		$P = .01$	
c	α	α/c	α	α/c	α	α/c
1	.0001	.0001	.0010	.001	.010	.010
2	.0142	.0071	.0454	.023	.149	.075
3	.0862	.0289	.191	.064	.436	.145
4	.232	.0580	.429	.107	.823	.206
5	.444	.0888	.739	.148	1.28	.256
6	.714	.119	1.11	.185	1.79	.298
7	1.03	.147	1.52	.217	2.33	.333
8	1.39	.174	1.97	.246	2.91	.364
9	1.78	.198	2.45	.272	3.51	.390
10	2.20	.220	2.96	.296	4.13	.413
12	3.11	.259	4.04	.337	5.43	.452
14	4.11	.293	5.20	.371	6.78	.484
16	5.17	.323	6.41	.401	8.18	.512
18	6.28	.349	7.66	.426	9.62	.534
20	7.44	.372	8.96	.448	11.1	.555
30	13.7	.457	15.9	.530	18.7	.623
40	20.6	.515	23.3	.582	26.8	.670
60	35.4	.590	38.0	.648	43.5	.725
80	50.9	.636	55.2	.690	60.7	.759
100	67.0	.670	71.9	.719	78.2	.782
140	100.2	.716	106.3	.760	114.0	.814
180	134.3	.746	141.4	.786	150.3	.835
200	151.6	.758	159	.795	169	.845
300	240	.800	249	.830	261	.870
400	330	.825	341	.852	355	.887
500	421	.842	434	.868	450	.900
600	513	.855	527	.878	545	.908
700	606	.866	621	.887	640	.914
800	699	.874	715	.894	736	.920
900	793	.881	810	.900	832	.924
1000	887	.887	905	.905	928	.928

BOOK REVIEWS.

W. H. BUSSEY, Chairman of the Committee.

Academic Algebra. By GEORGE WENTWORTH and DAVID EUGENE SMITH. Ginn & Co., Boston, 1913. iv + 442 pages. $1.20.

This book is the second of the Wentworth-Smith series. In it the authors have attempted to provide a high school course which shall cover the topics

named in the various curricula suggested by educational associations. The topics considered valuable but not essential are placed in an appendix. More than this, the authors have sought, by means of the practical problems, to give proper preparation to those who are fitting themselves for a trade. The noteworthy features of the book are the early and simple introduction of graphs with a table of squares and cubes at the end of the book to facilitate computation, the large number of oral problems under each topic and the cumulative reviews at the end of the book. There is also a very brief history of Algebra.

R. R. SHUMWAY.

Higher Algebra. By HERBERT E. HAWKES. Ginn and Company, Boston, 1913. vi + 222 pages. $1.25.

This algebra is an admirable rearrangement and revision of selected chapters from the *Advanced Algebra* published by the same author eight years ago. Pedagogically and typographically the book marks a distinct advance over the earlier work.

As a relatively large number of texts are upon the market dealing with high-school algebra and as more bid fair to appear, it would be highly desirable for the publishers to agree upon a somewhat precise definition of the terms, higher algebra, advanced algebra, college algebra, elementary algebra, and algebra. Our terminology, unfortunately somewhat fixed, seems distinctly inferior to that employed in Europe. A foreign mathematician who chances upon an American "higher algebra" with the content of this one would doubtless be led to inquire into the nature of our "lower algebra." On the other hand, the appearance of our books is decidedly superior, though even this is not an unmixed blessing, for the complete explanations accompanied by numerous diagrams bear mute witness to the fact that many high school teachers require all this paraphernalia to present the subject. Even with the best teachers such complete texts as our American publishers and authors are presenting have their disadvantages, for the pupil is encouraged to believe that the teacher's explanation is of secondary importance. A desirable innovation for test purposes would be a text-book on algebra consisting only of a few formulas, possibly those commonly put in heavy type, together with long sets of exercises. Pupils would be compelled to attend closely to class demonstrations and the teacher would become something more than a commentator.

Eleven chapters are included in this work together with four-place logarithms and tables for the extraction of square and cube roots. The chapter headings are as follows: Introductory review, functions and their graphs, quadratic equations, inequalities, complex numbers, theory of equations, permutations, combinations, and probability, determinants, partial fractions, logarithms, infinite series.

Only minor points would seem to require criticism. The paragraph, page 48, on the "reduced form" $x^2 + px + q = 0$ of the quadratic $ax^2 + bx + c = 0$ seems unnecessary and undesirable. Furthermore, "reduce" is twice used in the introductory discussion of this passage in a different sense from that of the

caption. The omission of any discussion of even the simplest types of simultaneous quadratics is unfortunate in a text explicitly designed for "the student who will continue his mathematics as far as the calculus." In analytic geometry simultaneous quadratics occur as frequently as any topic of algebra. The extension of Horner's method to the computation of larger roots by approximating first to hundreds, then tens, then units, is desirable. No hint of this is given in the text nor is any problem presented which suggests this extension.

Only one historical note is found in the text and that involves a disputed point. Tartaglia arrived at the solution of the cubic before Cardan, as the latter stated in his published work. However Scipio Ferro (died 1526), also cited by Cardan, was prior to both in the solution of the cubic of the form, $x^3 + ax = b$.

L. C. Karpinski.

PROBLEMS AND QUESTIONS.

B. F. Finkel, Chairman of the Committee.

PROBLEMS FOR SOLUTION.

ALGEBRA.

388. Proposed by JOSEPH V. COLLINS, Stevens Point, Wis.

On a certain typewriter there is a double scale as follows:

80........70........60........50........40........30........20........10........ 0
1 .... 5........10........15........20........25........30........35........40

It is used to locate headings in the middle of the page. To space a heading one sets the machine at the right stop and with the spacer counts out the number of letters and spaces in the heading. To the reading on the 40 scale where the carriage stops is added the reading of the right stop on the same scale. This number is the one on which to set the carriage pointer on the 80 scale to begin the heading. Show by algebra that the method is correct.

389. Proposed by W. W. BEMAN, Univ. of Michigan.

If $e^{e^x} = 1 + a_1x + a_2x^2 + a_3x^3 \cdots$, prove that

$$na_n = \Sigma_{k=1}^{k=n} \frac{1}{(k-1)!} a_{n-k}, \qquad \text{or} \qquad n!\,a_n = \Sigma_{k=1}^{k=n} \frac{(n-1)!}{(k-1)!} a_{n-k},$$

which latter form lends itself more readily to computation.

390. Proposed by E. B. ESCOTT, University of Michigan.

Sum the series,

$$\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{2}{4} + \tfrac{3}{8} + \tfrac{5}{16} + \tfrac{8}{32} + \cdots,$$

where each numerator is the sum of the two preceding and the denominators are in geometrical progression.

391. Proposed by C. N. SCHMALL, New York, N. Y.

Show that the roots of the quadratic

$$ax^2 + 2bx + c = 0$$

are imaginary if a, b, c, are in harmonic progression and have the same sign.

GEOMETRY.

417. Proposed by R. P. BAKER, University of Iowa.

Enumerate the points in which the twelve dihedral bisector planes of a tetrahedron meet, find their multiplicity and account for the 220 points which 12 planes in general determine.

418. Proposed by MRS. H. E. TREFETHEN, Waterville, Maine.

The difference of the squares of the two interior diagonals of a cyclic quadrilateral is to twice their rectangle as the distance between their mid-points is to the third diagonal.

419. Proposed by S. LEFSCHETZ, University of Nebraska.

Given a circle and a tangent to it. To find a point on its circumference such that the sum of its distances to the tangent and its point of contact shall be equal to a given length.

(Rouché et de Comberousse: *Géométrie.*)

CALCULUS.

339. Proposed by T. H. GRONWALL, Chicago, Ill.

To show that for any real value of x

$$\left|\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)\right| \leq \frac{1}{n+1}, \qquad \text{and} \qquad \left|\frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right)\right| \leq \frac{1}{n+1}.$$

340. Proposed by C. N. SCHMALL, New York, N. Y.

A pencil of parallel rays of light is incident upon a lens whose faces have the radii r_1, r_2, respectively. Show that the distance of the principal focus from the center of the first face of the lens will be a maximum or a minimum when

$$\frac{r_1}{r_2} = \frac{(\mu - 1)^{1/2}}{1 + (\mu - 1)^{1/2}},$$

where μ has its usual meaning.

341. Proposed by E. B. ESCOTT, University of Michigan.

Find an expression for the volume of a barrel in terms of the length l, the bung diameter a, and the head diameter b, also an approximate expression when a and b are nearly equal.

MECHANICS.

271. Proposed by V. M. SPUNAR, Chicago, Ill.

Find the center of gravity of the volume formed by the revolution around the x-axis of the area of the curve $y^4 - axy^2 + x^4 = 0$.

272. Proposed by W. J. GREENSTREET, Editor Mathematical Gazette, England.

Around a smooth fixed circular pulley is wound a massless inextensible string, and straight portions go to two free ends A and B to which masses are fastened. The mass at A is initially projected perpendicular to the string while the other is initially at rest. The length of the straight portion to the first mass is initially l and subsequently is r. Find the velocity of the second mass at that moment.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

187. Proposed by L. E. DICKSON, University of Chicago.

Find an amicable number triple by solving one of the equations (other than the last) in the MONTHLY, March, 1913, page 92. Note that a solution a is to be excluded if not prime to the numbers in the same line.

188. Proposed by ARTEMAS MARTIN, Washington, D. C.

Find rational values for v, w and x that will satisfy simultaneously the conditions

$$(m^2 + n^2)(v^2 + w^2 + x^2)^2 - 4m^2n^2v^2 + m^2n^2(m^2 + n^2) = 0, \tag{1}$$

$$(m^2 + n^2)(v^2 + w^2 + x^2)^2 - 4m^2n^2w^2 + m^2n^2(m^2 + n^2) = 0, \tag{2}$$

$$(m^2 + n^2)(v^2 + w^2 + x^2) - 4m^2n^2x^2 + m^2n^2(m^2 + n^2) = 0, \tag{3}$$

m and n being known quantities.

189. Proposed by W. E. HEAL, Washington, D. C.

Develop a formula for the value of x in the equation $p^x + q^x = s^x$, p, q, and s being integral numbers.

SOLUTIONS OF PROBLEMS.

GEOMETRY.

392. Proposed by V. M. SPUNAR, Chicago, Ill.

A tangent to a curve at any point P cuts the tangent and the normal at a fixed point O in the points M and N and the rectangle $OMP'N$ is completed. Find the curve which is such that the triangle formed by the tangents at any three points P, Q, R, is equal to the triangle formed by the corresponding points, P', Q', R'.

SOLUTION BY THE PROPOSER.

For the sake of simplicity assume the fixed point O to be the origin and take the tangent and normal to the curve at that point as the co-ordinates of reference. Let the three tangents to the curve be

$$b_1x + a_1y - a_1b_1 = 0, \tag{1}$$

$$b_2x + a_2y - a_2b_2 = 0, \tag{2}$$

$$b_3x + a_3y - a_3b_3 = 0, \tag{3}$$

where (a_i, b_i) are the intercepts of the tangents on the axes. Then, for the double area of the triangle formed by the three tangent lines, we have

$$2\Delta = \frac{[a_1a_2b_3(b_1 - b_2) + a_2a_3b_1(b_2 - b_3) + a_3a_1b_2(b_3 - b_1)]^2}{(a_1b_2 - a_2b_1)(a_1b_3 - a_3b_1)(a_2b_3 - a_3b_2)}. \tag{4}$$

Again, the double area of the triangle formed by the three points (a_1, b_1), (a_2, b_2), (a_3, b_3) is $2\Delta' = b_1(a_2 - a_3) + b_2(a_3 - a_1) + b_3(a_1 - a_2)$. (5)

To be more general, let

$$\Delta = m\Delta'; \tag{6}$$

or

$$\frac{[a_1a_2b_3(b_1 - b_2) + a_2a_3b_1(b_2 - b_3) + a_3a_1b_2(b_3 - b_1)]^2}{(a_1b_2 - a_2b_1)(a_1b_3 - a_3b_1)(a_2b_3 - a_3b_2)}$$
$$= m[b_1(a_2 - a_3) + b_2(a_3 - a_1) + b_3(a_1 - a_2)]. \tag{7}$$

Let the first two tangents be constant and the third one variable with intercepts of the form

$$a_3 = x - y(dx/dy) \quad \text{and} \quad b_3 = y - x(dy/dx)$$

so that

$$b_3 = -a_3p, \quad \text{where} \quad p = dy/dx.$$

Then, substituting a_3 and b_3 from the last expressions in equation (7) after some easy reductions, we get

$$p^4Qx - p^3(C^2x^2 - Lx + Py + N) - p^2(2C^2xy + Mx - Ly - S)$$
$$- p(C^2y^2 + My - Rx + P) - Ry = 0 = \phi(x, y, p), \tag{8}$$

where the following abbreviations are used:

$$a = a_1a_2, \quad b = b_1b_2, \quad A = a_2 - a_1, \quad B = b_2 - b_1,$$

$$C = a_1b_2 - a_2b_1, \quad D = a_1b_2 + a_2b_1, \quad L = C[ADm + (m+2)aB],$$

$$M = C[BDm + (m+2)bA], \quad N = a(C^2m + aB^2), \quad P = b(C^2m + bA^2),$$

$$Q = maAC, \quad R = mbBC, \quad S = (mC^2D + 2abAB).$$

If we look upon (8) as an algebraical equation in $p = dy/dx$ which has 4 roots p_1, p_2, p_3, p_4, these being functions of x and y, then we are led to the conclusion that through any point in the plane there are *four* directions satisfying the condition proposed. We have, therefore, *four* curves in question.

For the present purpose the equation (8) in p may be considered irreducible. Then, if a singular solution of this equation exists, it must simultaneously satisfy the equations

$$\varphi = 0, \quad \frac{\partial\varphi}{\partial p} = 0, \quad \frac{\partial\varphi}{\partial x} + p\frac{\partial\varphi}{\partial y} = 0.$$

The last equation is *identically zero* in the present case, and we have

$$\frac{d\varphi}{dp} = 4p^3Qx - 3p^2(C^2x^2 - Lx + Py + N) - 2p(2C^2xy + Mx - Ly - S) - (C^2y^2 + My - Rx + P) = 0. \tag{9}$$

Equations (8) and (9), therefore, represent, in parametric form, the singular solution of the differential equation (8), which is, then, the algebraic curve in question.

411. Proposed by C. N. SCHMALL, New York City.

ABCD is a rectangle of known sides. *BC* being produced indefinitely, it is required to draw a straight line from *A* cutting *CD* and *BC* in *X* and *Y*, respectively, so that the intercept *XY* may be equal to a given straight line. (Unsolved in *Educational Times*.)

II. SOLUTION BY THE PROPOSER.

We shall assume that the given rectangle is a square, the problem thus being a special case of problem 382, proposed by R. C. Archibald in the May (1911) number of the MONTHLY.

CONSTRUCTION: Along AB produced lay off AE equal to the *given length.* Draw ED. Prolong AD to K so that $DK = DE$. On AK as a diameter (centre O) describe a semi-circle cutting BC produced in Y. Draw AY cutting DC in X. Then AXY is the line required; *i. e.*, the intercept XY is of the given length.

PROOF: Draw YH perpendicular to AK. Draw XK and YK. Then the right triangles ADX and YHK are equal in all respects (congruent). For the angles DAX and HYK are equal, and $DA = DC = HY$.

Hence

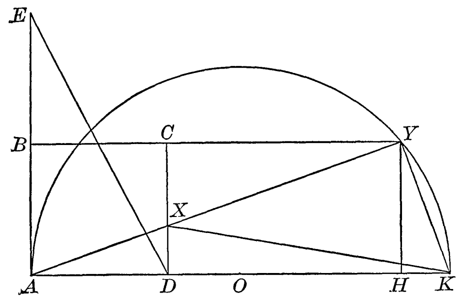
$$AX = YK. \tag{1}$$

Now

$$\overline{XY}^2 + \overline{YK}^2 = \overline{XK}^2 = \overline{XD}^2 + \overline{DK}^2 = \overline{XD}^2 + \overline{DE}^2 \text{ (by constn.)} \tag{2}$$

$$= \overline{XD}^2 + \overline{AD}^2 + \overline{AE}^2 = \overline{AX}^2 + \overline{AE}^2 = \overline{YK}^2 + \overline{AE}^2, \text{ by (1)}.$$

Hence, $\overline{XY}^2 = \overline{AE}^2$ and, therefore, $XY = AE =$ the given length.



Also solved by J. Scheffer whose conclusion agrees with that of Professor Shively published in the April number.

H. C. Feemster sent in a solution of 409 too late for notice in April number.

MECHANICS.

250. Proposed by C. N. SCHMALL, New York City.

A smooth circular table in surrounded by a smooth vertical rim. A ball of elasticity e is projected from a point at the rim in a line making an angle θ with the radius through that point. Show that the ball will return to the starting point after the second impact if

$$\tan\theta = \sqrt{\frac{e^3}{e^2+e+1}}.$$

Solution by the Proposer.

Let ABC be the path of the ball and O the center of the table. Let the angles OAB, OBC, be θ, ϕ respectively. Then OBA and OCB are also θ, ϕ, respectively. Let OCA be ψ. Then if the ball returns to A after the second impact the angle CAO will also be ψ. The angles θ, ϕ, ψ, are the angles which the path of the ball makes with the radii passing through it initially and after the first and second impacts. We then have, by elementary principles of impact,

$$\cot\phi = e\cot\theta, \tag{1}$$

$$\cot\psi = e\cot\phi = e^2\cot\theta. \tag{2}$$

But $\theta + \phi + \psi = \pi/2$. Hence $\cot(\theta+\phi+\psi) = 0$.

That is,

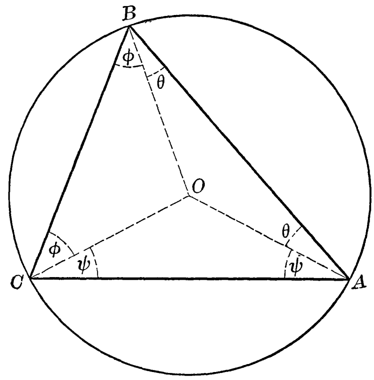
$$\frac{\cot\theta\cot\phi\cot\psi - \cot\theta - \cot\phi - \cot\psi}{\cot\theta\cot\phi + \cot\phi\cot\psi + \cot\psi\cot\theta - 1} = 0.$$

Hence $\cot\theta\cot\phi\cot\psi = \cot\theta + \cot\phi + \cot\psi$, and by (1) and (2),

$$e^3\cot^3\theta = \cot\theta\,(1 + e + e^2),$$

or,

$$\tan^2\theta = \frac{e^3}{e^2 + e + 1} \quad \text{and} \quad \tan\theta = \sqrt{\frac{e^3}{e^2 + e + 1}}.$$



NOTES AND NEWS.

Mr. G. W. Smith, of the University of Colorado, has been appointed assistant in mathematics at the University of Illinois.

Dr. B. F. Finkel, of Drury College, will have charge of the mathematics in the 1913 summer school of the University of Colorado.

The Monthly will not be published during the months of July and August. The next issue will appear early in September.

The April number of *The Monist* contains an article by the late Henry Poincaré on "The Relativity of Space."

The first two articles in the latest number of the *Bibliotheca Mathematica* (May 6, 1913) are by Professor L. C. Karpinski. Their titles are as follows: "Hindu numerals among the Arabs," and "The Quadripartitum numerorum of Johannes de Muris."

Mr. G. W. Hess, now fellow in mathematics at the University of Michigan, has accepted a call to Shurtleff College, Alton, Illinois, as head of the department of mathematics.

Mr. David M. Smith and Mr. David L. Stamy, graduate students at the University of Chicago, have been appointed instructors in mathematics at the Georgia School of Technology, Atlanta, Ga.

The following candidates for the doctorate in mathematics at the University of Chicago have been appointed to University instructorships: Miss Mildred L.

SANDERSON at the University of Wisconsin, MR. WILLIAM O. BEAL and MR. WILSON L. MISER at the University of Minnesota, and MR. CHARLES R. DINES at Northwestern University.

A book on "Graphical Methods" by CARL RUNGE has just been brought out by the Columbia University Press. It deals with graphical calculation, the graphical representation of functions of one or more independent variables, the graphical methods of the differential and integral calculus.

The prices of full sets of analytical cards, for the German and French editions of the Mathematical Encyklopädie, as issued at this date by the Library of Congress, are as follows: (1) Author set, that is, one copy of each card, German edition \$2.60; French edition \$1.60; (2) Dictionary catalog set, that is, enough cards for main and secondary entries, German edition \$4.08; French edition \$1.60.

PROFESSOR C. J. KEYSER, of Columbia University, recently completed a lecture tour arranged by the Sigma Xi Societies of the Universities of Iowa, Minnesota, and Nebraska. The subject of his lecture was "Concerning the figure and the dimensions of the Universe of space."

Marked activity in researches on the history of oriental mathematics is shown by the recent publication of two books which in a way supplement each other, the first by YOSHIO MIKAMI on "The Development of Mathematics in China and Japan" (Leipsig, B. G. Teubner), the second by DAVID EUGENE SMITH and YOSHIO MIKAMI on "The History of Japanese Mathematics" (Chicago, Open Court Publishing Company).

PROFESSOR SAUL EPSTEEN, professor of engineering mathematics at the University of Colorado, has been granted a sabbatic leave of absence. He will spend the year at the state capitol in Denver as commissioner of insurance.

During 1913–14 there will be given at the Ohio State University a "Course for Teachers of Mathematics," devoted to the pedagogy and the history of mathematics. Practice will be given in the presentation of typical subjects and exercises in algebra and geometry and an endeavor will be made to give the student a modern point of view of the field of mathematics. Another course on "Mathematics of Statistics, Finance and Insurance" is offered as an introduction to the mathematical principles of statistical methods, investments and insurance.

The Society of the Sigma Xi which is devoted to the interests of research in the different sciences has begun the publication of a periodical under the title *Sigma Xi Quarterly*. The editorial committee is composed of McKEEN CATTELL, DAYTON C. MILLER, H. B. WARD, and S. W. WILLISTON. The first number is dated March, 1913, and is largely devoted to the proceedings of the fourteenth convention, held at Cleveland on January 2, 1913.

The 1912–13 edition of *Who's Who in America* includes the names of about 175 members of the American Mathematical Society. It seems therefore safe to conclude that less than one per cent. of the 18,794 sketches of "conspicuous

living Americans" appearing in this work relate to men who have gained distinction by their mathematical work. About one fourth of the names of the present American Mathematical Society appear in *Who's Who*. Viewed from the standpoint of mathematical attainments and proficiency the selection could be much improved.

Dr. James W. Glover, professor of mathematics and insurance, at the University of Michigan, has been appointed expert special agent of the Bureau of the Census to supervise the preparation of a special volume on vital statistics. Extensive mortality tables are to be prepared, based on the population and vital statistics of the United States. Dr. Glover has also been appointed collaborator to the Office of Public Roads in the Department of Agriculture to assist in the preparation of several bulletins on the various methods of issuing and financing public highway bonds. He will also offer a course of lectures this summer at the University of Oregon summer session on the applications of mathematics to finance, insurance, and statistics.

A reader of the Monthly in former years writes that it has become toc abstruse to be read by men of moderate mathematical attainments. An examination of the first five numbers of the present volume will show that 138 pages out of a total of 168 have been devoted to notes, reviews, solution of problems, pedagogical and historical articles and very elementary mathematical papers, while the remaining 30 pages have been given to mathematical articles of a more technical (though elementary) character. It would seem that this is a fair division of subject matter. In this connection see the article in this issue on "Incentives to Mathematical Activity."

"An Introduction to the Mathematical Theory of Heat Conduction with Engineering and Geological Applications" by L. R. Ingersoll and O. J. Zobel, of the University of Wisconsin, is a new book just issued by Ginn & Company. This able text will interest teachers desiring to offer graduate or advanced undergraduate courses on heat conduction. It lays special emphasis upon practical applications but does not enter very extensively into recondite mathematical discussion.

A special commemorative issue of the *Mathematical Gazette*, in honor of the founder, E. M. Langley, and the present editor, W. J. Greenstreet, was issued in January, 1913, and was edited by A. Lodge. It begins with brief histories of the Association for the Improvement of Geometrical Teaching and of its successor, the Mathematical Association, an organization of teachers and students of elementary mathematics. This association has over 700 members. About three fourths of these live in England, while the remainder are distributed over a number of countries. There are eight in Canada and twenty-one in the United States. Sir George Greenhill was elected President for the years 1913 and 1914.

Professor E. W. Hobson's Presidential Address before the Mathematical Association was delivered on January 8, 1913, and was devoted to a consideration

of constructions by means of the compass. Professor Hobson laid stress on two fundamental views of geometry. "On the practical side, geometry is a physical science in which the objects dealt with—points, straight lines, circles, etc.—are physical objects, the relations between which are to be ascertained and described with a view to dealing with actual spatial relations with an accuracy sufficient for all practical purposes. On the theoretical side, geometry is an abstract science." This address appeared in the March, 1913, number of the *Mathematical Gazette.*

The twentieth summer meeting and seventh colloquium of the American Mathematical Society will be held at the University of Wisconsin, Madison, Wis., during the week beginning Monday, September 8, 1913. The first two days will be devoted to the regular sessions for the presentation of papers. The colloquium will open on Wednesday morning and will close on Saturday morning. Courses of lectures will be given as follows:

PROFESSOR L. E. DICKSON: "Certain aspects of a general theory of invariants, with special consideration of modular invariants and modular geometry." Attention will be limited to the ordinary theory of numbers, and no reference will be made to the general finite field.

PROFESSOR W. F. OSGOOD: "Topics in the theory of analytic functions of several complex variables." The lectures will attempt to give a brief survey of what has been accomplished in the study of some of the more important problems of this branch of analysis.

The Junior Mathematical Club of the University of Chicago has for several years been conducted by the younger graduate students as an adjunct to the Graduate Club and Seminar which is conducted by the faculty and more advanced graduate students. The Junior Club meets fortnightly, and the following papers have been presented during the present year:

"The purpose and work of the club." PROFESSOR H. E. SLAUGHT.

"On the notion of function." PROFESSOR E. H. MOORE.

"Fermat numbers." MR. C. R. DINES.

"On the acoustic figures of a rectangular vibrating membrane." MISS MARTHA MACDONALD.

"On a particular case of an elastic pendulum." MR. YOSHIO ISHIDA.

"An introduction to a function theoretic theory of invariants." PROFESSOR L. E. DICKSON.

"On defining the elliptic functions." MR. W. MISER.

"Some higher plane curves." MISS MARY E. WELLS.

"The radius vector as a function of the time in the hyperbolic case of the problem of two bodies." MR. S. M. SEWALL.

"A problem in Abelian integrals." MR. W. C. VERNON.

"On the projective geometry of the cycloid." MR. J. O. HASSLER.

"On Green's functions." MR. W. C. WESTER.

The University of Edinburgh has resolved upon the establishment of a laboratory for practical instruction in numerical, graphical, and mechanical calculation and analysis, as required in the applied mathematical sciences, and for research in connection with the mathematical department. The courses, under the direction of PROFESSOR E. T. WHITTAKER, embrace the following topics: Differences and interpolation, computations with tables of logarithms, log sines, natural sines, products, quarter-squares, etc., numerical solution of trigonometric problems; Controls for checking accuracy of computations, design of computing-forms; Method of least squares, numerical solutions of systems of linear equations, numerical evaluation of determinants; Curve-fitting, calculation of correlation-coefficients; Analysis of a function into sine and cosine terms (practical Fourier analysis); Analysis for the discovery of periodic constituents in a function (periodogram analysis), practical spherical harmonic analysis, other methods of analysis of functions empirically given; Construction of curves and surfaces, linkages, roulettes, projections, photogrammetry, map-making, graphic solution of numerical equations, graphic and mechanical solution of problems in spherical trigonometry, nomography, applications of triangle of vectors; Use of instruments employed in calculation, especially slide rules, arithmometers, planimeters, integraphs, and harmonic analysers; Numerical evaluation of definite integrals; Numerical solution of differential equations; Numerical evaluation of roots, etc., of transcendental functions; Calculations performed with elliptic functions, arcs on spheroids, etc.; Formation and use of tables of Legendre's and Bessel's functions, the gamma function, error-function, and other transcendental functions; Construction of tables of new functions, and functions not previously tabulated, including automorphic functions, and the parabolic-cylinder and elliptic-cylinder functions.

An important mathematical activity, which should have been mentioned in the leading article of this issue, is the work of the Committee of Fifteen, under the chairmanship of PROFESSOR E. V. HUNTINGTON, of Harvard University, representing the common interests of mathematicians and engineers as voiced in the joint meetings held at Chicago in 1907, under the auspices of the Chicago Section of the American Mathematical Society. The report of this Committee is a syllabus, in book form, of courses in mathematics for students of engineering, published by the Society for the Promotion of Engineering Education. (See the advertisement in this issue.)

PROFESSOR CAJORI'S "History of Logarithms" will be concluded in the September issue. A limited number of copies of the MONTHLY containing this history can be supplied to new subscribers in the order of application. The readers of the MONTHLY will confer a favor upon their friends by calling this information to their attention. Subscriptions should be sent to the Treasurer, B. F. Finkel, Springfield, Missouri.

H. E. SLAUGHT,
Managing Editor.

IMPORTANT ANNOUNCEMENT

TO

ALL TEACHERS OF MATHEMATICS

AND

OTHERS INTERESTED IN MATHEMATICAL PROGRESS

THE AMERICAN MATHEMATICAL MONTHLY, since its reorganization in January, 1913, has endeavored to fulfill its mission as "A JOURNAL FOR TEACHERS OF MATHEMATICS IN THE COLLEGIATE AND ADVANCED SECONDARY FIELDS."

A selection from the Tables of Contents of the first six numbers includes articles on—

The History of Mathematics, such as the following:

"History of the Exponential and Logarithmic Concepts," by PROFESSOR FLORIAN CAJORI, of Colorado College;

"The Foundation Period in the History of Group Theory," by JOSEPHINE BURNS, Graduate Student at the University of Illinois;

"Errors in the Literature on Groups of Finite Order," by PROFESSOR G. A. MILLER, University of Illinois.

Pedagogical Considerations, such as the following:

The "Foreword" concerning Collegiate Mathematics, by PROFESSOR E. R. HEDRICK, University of Missouri;

"Mathematical Literature for High Schools," by PROFESSOR G. A. MILLER;

"Minimum Courses in Engineering Mathematics," by PROFESSOR SAUL EPSTEEN, University of Colorado;

"Incentives to Mathematical Activity," by PROFESSOR H. E. SLAUGHT, University of Chicago.

General Mathematical Information, such as the following:

"The Third Cleveland Meeting of the American Association for the Advancement of Science," by PROFESSOR G. A. MILLER;

"Western Meetings of Mathematicians," by PROFESSOR H. E. SLAUGHT;

"Notes and News" of events pertaining to mathematics, under the direction of a committee of which PROFESSOR FLORIAN CAJORI is chairman;

"Book Reviews" and announcements of new books in mathematics, under the direction of a committee of which PROFESSOR W. H. BUSSEY, University of Minnesota, is chairman.

Topics Involving a Minimum of Technical Treatment, such as the following:

"Maximum Parcels under the New Parcel Post Law," by PROFESSOR W. H. BUSSEY;

"Precise Measurements with a Steel Tape," by PROFESSOR G. R. DEAN, Missouri School of Mines;

"A Direct Definition of Logarithmic Derivative," by PROFESSOR E. R. HEDRICK;

"A Simple Formula for the Angle Between Two Planes," by PROFESSOR E. V. HUNTINGTON, Harvard University;

"Two Geometrical Applications of the Method of Least Squares," by PROFESSOR J. L. COOLIDGE, Harvard University;

"Problems Proposed and Solved," under the direction of a committee of which PROFESSOR B. F. FINKEL, Drury College, is chairman.

Topics Involving Somewhat More Technical Treatment, designed to stimulate mathematical activity on the part of ambitious students and teachers. Such articles have occupied only about one-sixth of the entire space; for example, such as the following:

"The Remainder Term in a Certain Development of $F(a+x)$," by PROFESSOR R. D. CARMICHAEL, Indiana University;

"A Geometric Interpretation of the Function F in Hyperbolic Orbits," by PROFESSOR W. O. BEAL, Illinois College;

"Certain Theorems in the Theory of Quadratic Residues," by PROFESSOR D. N. LEHMER, University of California;

"Some Inverse Problems in the Calculus of Variations," by DR. E. J. MILES, Yale University;

"Amicable Number Triples," by PROFESSOR L. E. DICKSON, University of Chicago.

Professor Cajori's " History of Logarithms " has run serially from the beginning and will have covered about 60 pages when concluded in the September issue. This is not only a contribution of great value but it is, at the same time, an extremely fascinating piece of reading. It will not be obtainable elsewhere after the present limited number of extra copies is exhausted, and these will be supplied only to new subscribers to the MONTHLY.

The Subscription Price of the Monthly is only TWO DOLLARS ($2.50 in foreign countries) per volume of ten numbers containing about 350 pages. Subscriptions should be sent to the Treasurer, B. F. Finkel, Springfield, Missouri.

VOLUME XX SEPTEMBER, 1913 NUMBER 7

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BY FLORIAN CAJORI, Colorado College.

V. GENERALIZATIONS AND REFINEMENTS EFFECTED DURING THE NINETEENTH CENTURY.

Continued from page 182.

UNIFORMIZATION.

Without the use of the logarithmic function, the interpretation of the power $z^y = w$ becomes complicated even in the simple case when z is negative and y is real. For integral values of y, w then becomes alternately $+$ and $-$; while for rational values m/n, w frequently becomes imaginary, and the pairs of values (z, w), all taken together, yield no continuous curve. At this stage the logarithmic function is introduced to great advantage through the relation $z^{m/n} = e^{m/n \log z}$. Here the n-valued expression $z^{m/n}$ becomes a one-valued function of $\log z$. We say that $z^y = w$ has been *uniformized.* That is w, which in the case when $y=m/n$ is an n-valued function of z, becomes a one-valued function of $\log z$. The function $\log z$ is a *uniformizing function.* As we look back, we see that the first traces of *uniformization* occurred in the eighteenth century, but not until recently has the process been formally introduced into analysis as a *conscious method.*[1] When in $z^y = w$, y is a real and irrational constant, or when y is a complex constant and z is a complex variable, then w is an infinitely many-valued function of z. In these cases the uniformization $w = z^y = e^{y \log z}$ is still more marked, since w descends from an infinitely many-valued function of z to a one-valued function of $\log z$.

PRINCIPAL VALUES OF POWERS AND LOGARITHMS. PROPOSED NOTATIONS.

Few researches on our topic carried on during the nineteenth century are of greater importance than those introducing the idea of principal values and sug-

[1] See F. Klein, *Elementarmath. v. höheren Standpunkte aus,* T. I, Leipzig, 1908, pp. 296, 352. Consult also E. T. Whittaker, *Modern Analysis,* Cambridge, 1902, p. 338.

gesting suitable notations. A leader in this movement was the great French mathematician, Augustin-Louis Cauchy.

In the *Cours d'analyse*,[1] Cauchy introduced notations designed to facilitate operations with general powers, roots, and logarithms. To designate any (*i. e.*, all) of the values, he used $((\alpha + \beta\sqrt{-1}))^{1/n}$, n being an integer. Taking m and n to be positive integers, relatively prime, and using trigonometry, he finds the values of $((\pm 1))^n$, $((\pm 1))^{m/n}$, $((\pm 1))^{-m/n}$, as well as the different powers of $\alpha + \beta\sqrt{-1}$. He suggests the use of double parentheses for inverse trigonometric functions as well, thus, arc sin $((a))$. For the principal values he uses the ordinary symbols $(\pm a)^{m/n}$, arc sin (a). By $L((x))$ he expresses any (*i. e.*, all) of the logarithms of the real or complex number x, to the base A. When the base is e, he writes $l((x))$; $l(2)$ signifying the real, natural logarithm of the number. Thus he writes, when α is real and negative, $l((\alpha)) = l(-\alpha) + l((-1)) = l(-\alpha) \pm (2k+1)\pi\sqrt{-1}$, $k = 0, \pm 1, \cdots$. If $\alpha + \beta\sqrt{-1} = \rho(\cos\theta + \sqrt{-1}\sin\theta)$, then Cauchy[2] writes $l((\alpha + \beta\sqrt{-1})) = l(\rho) + \theta\sqrt{-1} + l((1))$, θ receiving some one value consistent with the given values of its sine and cosine. On the other hand, $l(\alpha + \beta\sqrt{-1}) = l(\rho) + \theta\sqrt{-1}$, where θ is some one of the infinite number of arcs. He finds that $L(x) + L(y) + \cdots = L(xy\cdots)$ holds only when the real part of each complex number $x, y, \cdots$ is positive, that the formula $L(x^\mu) = \mu L(x)$ holds only when α in $x = \alpha + \beta\sqrt{-1}$ is positive, and $\mu \times$ arc tan α/β lies between $-\pi/2$ and $\pi/2$, μ being real. This notation is explained also in the beginning of Cauchy's *Exercices de mathématiques*, Paris, 1826. Later Cauchy found that his definition for $l(x)$ was unsatisfactory from the standpoint of the continuity of the function $l(x)$. Hence in 1846 he suggested the following modification of the definition:[3] $l(x) = l(\rho) + \theta\sqrt{-1}$, where θ is permitted to vary between $-\pi$ and $+\pi$, $-\pi$ itself being excluded, but $+\pi$ included. In the same way $(x)^m$ or x^m was defined as $\rho^m e^{m\theta\sqrt{-1}}$, with the same restrictions on θ. Similar recommendations were made at about the same time by E. G. Björling in Sweden.[4]

Other authors chose different restrictions for the argument θ. Thus John Warren in 1829 adopted the convention $0 < \theta < 2\pi$.[5]

In 1847 A. Cayley[6] modified somewhat Cauchy's early definition of principal values. He assumes them to be $(x + iy)^m = \rho^m e^{im\theta}$ when x is positive, and $(x + iy)^m = \rho^m e^{im(\theta\pm\pi)}$ when x is negative; θ lying between $+\pi/2$ and $-\pi/2$, and the $+$ or $-$ being taken for $\pm\pi$ according as y is $+$ or $-$. For logarithms Cayley gives as principal values, log $(x + iy) = \log\rho + i\theta$ when x is positive, and log $(x + iy) = \log\rho + i(\theta \pm \pi)$ when x is negative. He states that these principal values satisfy the equations $(x + iy)^m(x' + iy')^m = [(x+iy)(x'+iy')]^m$

[1] A. Cauchy, *Cours d'analyse*, Paris, 1821, Chap. VII, § I.
[2] Cauchy, *op. cit.*, Chap. IX.
[3] *Journal de mathématiques pur. et appl.*, T. 1846, p. 326.
[4] *Grunert's Archiv*, Vol. 21, 1853, p. 2.
[5] Warren, *Phil. Trans.*, London, 1829, Vol. 119, p. 339.
[6] A. Cayley, *Coll. Math. Papers*, Vol. I, Cambridge, 1889, p. 309.

and $\log(x + iy) + \log(x' + iy') = \log[(x + iy)(x' + iy')]$. In a second paper[1] of the year 1856 he states that this last equation is not always satisfied and he investigates the cases of failure. In 1869 was published an article by Cayley, *On Logarithms of Imaginary Quantities*,[2] in which he treats of the principal or "selected" values, adopting $-\pi$ and $+\pi$ as the angular limits.

The German term for principal value is *Hauptwert*, a term used by Karl Weierstrass in his lectures on Abelian transcendents delivered about 1875 in Berlin. Cauchy had taken, as the principal value of $\sqrt{a + bi}$, the value whose second coördinate has the same sign as the second coördinate b, in $a + bi$. Weierstrass modified this definition, so that the principal value of $\sqrt{a + bi}$ is the "positive"[3] value of the square root, provided that $a + bi$ is itself "positive." Weierstrass calls $a + bi$ "positive," when a is positive or, if $a = 0$, when b is positive. If $a + bi$ is negative, then the principal value of its square root shall be $i\sqrt{-a - bi}$, where $\sqrt{-a - bi}$ is "positive."

The terminology and notation of powers, roots, and logarithms received further attention from Cauchy in 1846.[4] When the variables in a function are extended from real to imaginary values, the notation and the formulas used for real variables cannot be conserved, unless one adopts new conventions to fix the meaning of the notation for imaginaries. Since a number has an infinity of logarithms, one cannot represent them all by one and the same notation without introducing confusion into the calculus. He lets lx stand for that natural logarithm of x in which the coefficient of $\sqrt{-1}$ lies between $+\pi$ and $-\pi$, the lower limit being excluded. Using Lx for the logarithm in any system, he has $Lx = Le^{lx}$, $x^a = e^{alx}$, $x^y = e^{ylx}$. With the above restriction these functions are in general *continuous* functions of x for real and complex values of the variables. These values Cauchy called the *logarithme principale, puissance principale,*—he used similarly the term *racine principale*. Part of this terminology was first suggested by E. G. Björling.

The notation $((1))^x$ was used in 1845 by T. Wittstein of Hanover in a proof that every equation has a root.[5] It was used in 1856 by the Swiss, H. Kinkelin[6] and is used by O. Stolz and J. A. Gmeiner in their well-known work, *Theoretische Arithmetik*.[7]

As a counterpart to Cauchy's notation for the general power, G. Peano has introduced the notation $\sqrt[m]{*a}$ to designate all the values that the mth root of a may take.[8]

Cauchy discusses the limit toward which $(1 + z/m)^m$ converges, for increasing positive integral values of the variable m, when z may be a complex number.[9]

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[1] Cayley, *Coll. Math. Papers*, Vol. III, p. 209.

[2] *Op. cit.*, Vol. VI, p. 14.

[3] K. Weierstrass, *Math. Werke*, Vol. IV, Berlin, 1902, pp. 348–350.

[4] *Comptes Rendus*, T. XXIII, p. 271; also *Exercices d'Analyse et d. Phys. Math.*, IV, 1847, p. 253; *Œuvres*, 1. S., T. X, p. 75.

[5] *Grunert's Archiv*, Vol. 6, 1845, p. 230.

[6] Same journal, Vol. 27, 1856, pp. 304–315.

[7] *Theor. Arith.*, 2 Abth., Leipzig, 1902, p. 373.

[8] *Formulaire de mathématiques* publié par la "Rivista di matematica," T. I, Turin, 1895, p. 19.

[9] Cauchy, *Exercices d'analyse et de phys. math.*, T. IV, Paris, 1847, pp. 232–246.

This limiting value, which is shown to exist, lends itself to the definition of the exponential function e^z. It follows that $e^z \cdot e^{z'} = e^{z+z'}$. Cauchy proceeds to the generalized power, in which both base and exponent are complex, or *quantités géométriques* as Cauchy calls them here.[1] If z and u are both complex, then z^u needs definition; he does it by the equation $z^u = e^{ul(z)}$. This definition includes, as special cases, the less general developments previously discussed by Cauchy. He states that E. G. Björling has given the same definition of z^u and has introduced the term *puissance principale.* This term Cauchy adopts, only he attributes to the argument p in $z^u = r^n e^{upi}$ values $-\pi < p \gtreqless \pi$, while Björling set the limits $(\pi/2) + \pi > p \gtreqless (\pi/2) - \pi$. By Cauchy's scheme, two conjugate complex numbers raised to powers indicated by conjugate numbers, respectively, yield for principal values, complex quantities which are themselves conjugate.

While the descartian exponential notation is excellent in all ordinary writing and printing, it is unsatisfactory for complicated exponents. Changes in notation have been proposed by De Morgan at the end of his Calculus of Functions, in the *Encyclopædia Metropolitana* (1843), and also in an article in the *Cambridge Philosophical Transactions,*[2] and later by J. W. L. Glaisher.[3] De Morgan would print " a raised to the power $(a + bx)/(c + dx)$ " thus, $a \wedge \{(a + bx)/(c + dx)\}$. Glaisher likewise suggests the use of arrows which he prefers to a third notation that had occurred to him, namely, the printing of a^u as a Exp. u. This last notation is the one used by A. Cayley in his article " Function " in the *Encyclopædia Britannica,* 9th edition, 1879. He writes e^u as exp. u. This notation has found wide acceptance in works on the theory of functions.

Certain novelties in logarithmic notation were suggested in Germany, but failed of adoption. As early as 1786 Abel Bürja[4] indicated the logarithmic inverse of $2^3 = 8$ by the notation $\frac{8}{\overline{\overline{2}}} = 3$. In 1811 Rothe[5] used the symbols 8?2, which were adopted in 1823 by Martin Ohm. In 1860 Köpp advocated a third notation which cannot be represented with ordinary type. His suggestion was opposed by F. A. Stadnička of Prag. The need of a new notation for logarithmation is not evident; the usual " $\log_2 8$ " seems to satisfy ordinary needs.

CLASSIFICATION OF LOGARITHMIC SYSTEMS.

A curious classification of logarithmic systems of real positive numbers was based by Hessel, of the University of Marburg, upon the theorem that, if $a > \sqrt[e]{e}$, where $e = 2.718\cdots$ and $\sqrt[e]{e} = 1.4445\cdots$, then $a^x > x$ for all (real) values of x. In his first group of logarithmic systems each number is greater than its (tabular) logarithm, as must be true when the base $a > \sqrt[e]{e}$. In his second system the base $a = \sqrt[e]{e}$; here there is one number, namely e, which is

[1] *Op. cit.,* p. 255.
[2] *Cambr. Phil. Trans.,* Vol. XI, Pt. III, p. 450.
[3] *Messenger of Mathematics,* Vol. II, 1873, pp. 107–111.
[4] A. Bürja, *Algebrist,* Berlin und Libau, 1. Theil, p. 153.
[5] Drænert in Hoffmann's *Zeitschr. f. Math. u. Phys.,* Vol. 8, Leipzig, 1877, p. 265.
[6] Grunert's *Archiv,* Vol. 14, 1850, p. 97.

equal to its own logarithm, but all other numbers are smaller than their respective logarithms. In his third group of systems, in which each base $a < \sqrt[e]{e}$, some of the numbers are larger, others are smaller than their respective logarithms, while for one number there is an equality between the two.

A general and more fundamental classification was given in 1892 by Irving Stringham[1] of the University of California. It was modified somewhat by M.W. Haskell and Stringham,[2] and formulated by Stringham in his *Uniplanar Algebra*, San Francisco, 1893. In the Wessel-Argand diagram, complex numbers are represented by vectors drawn from the origin to points on some logarithmic spiral; the logarithms of these numbers are represented by vectors from the origin to points on a straight line. The base is the particular vector drawn to the spiral which corresponds to the vector 1 drawn to the line; the modulus is, in general, a complex number. Logarithmic systems are classified as *gonic* and *agonic* systems, according as the modulus is a complex number or a real number.

LOGARITHMS AS DIRECT FUNCTIONS.

That it is possible to approach logarithms, not from the viewpoint of inverse functions, but from the standpoint of direct functions, was recognized early in the eighteenth century. John Bernoulli I, in his correspondence with Leibniz, started from the differential equation $dx/x = -dx/-x$ and thence deduced $\log x = \log(-x)$. This operation presented the logarithmic function as an integral, but labored under the defect of not making the integral definite. Clear ideas on this subject were entertained by Gauss who in a letter of Dec. 18, 1811, addressed to Bessel,[3] set forth the significance of $\int_1^x dx/x$ in the Wessel-Argand plane as an infinitely many-valued function of a complex variable. This view has been presented in works on the theory of functions.[4]

In 1869 an article was published by Cayley, *On logarithms of imaginary quantities*[5] in which the logarithm is defined as a definite integral and the general power a^b is defined as the inverse of the logarithmic function. Much the same course was pursued in 1902 by J. W. Bradshaw.[6] A radical change is advocated by Felix Klein. After discussing the difficulties encountered in defining a logarithm by the equation $a^b = c$, he goes so far as to recommend that, for purposes of school room instruction, the logarithm be defined as an integral and be connected with the asymptotic areas of the equilateral hyperbola.[7] He declares that the proper source for the introduction of new functions is the quadrature of familiar curves. This program is elaborated more fully in an article by C. Frenzel.[8]

[1] *American Jour. of Math.*, Vol. XIV, 1892, pp. 187–194.

[2] *Bulletin of the N. Y. Math. Society*, Vol. II, 1893, pp. 164–170.

[3] Gauss, *Werke*, Bd. VIII, p. 90.

[4] H. Durège, *Theorie der Funktionen einer compl. veränd. Grösse*, Leipzig, 3. Aufl., 1882, Absch. V.

[5] Cayley, *Coll. Math. Papers*, Vol. VI, p. 14.

[6] *Annals of Math.*, 2d S., Vol. 4, 1902–3; pp. 51–62. See also W. F. Osgood, *Funktionentheorie*, Leipzig u. Berlin, 1907, pp. 487–498.

[7] F. Klein, *Elementarmathematik vom höheren Standpunkte aus.*, I, Leipzig, 1908, p. 332.

[8] *Zeitschrift f. math. u. naturwiss. Unterricht*, 44. Jahrg., 1913, pp. 1–13.

A purely analytical theory of the logarithmic function, divorced from geometry, was developed in 1891 by Ch. Méray of Dijon.[1] He considered the logarithmic function as the result of integration and arrived at the exponential function by solving for u the equation $\log u = x$.

TABLE OF CONTENTS.

History of the Exponential and Logarithmic Concepts.

Additions and Corrections.

Page 9, line 23; G. Eneström points out that Halley's definition was given earlier by N. Mercator in his *Logarithmo-Technia*, 1768, thus: *Est enim logarithmus nihil aliud, quam numerus ratiuncularum contentarum in ratione, quam absolutus quisque ad unitatem obtinet* See also our footnote (2), page 9.

Page 13, line 2 up; *for* 1644 *read* 1649.

Page 44, equation (1) should read $i\varphi = \log(i \sin \varphi + \cos \varphi)$. Likewise, each ψ on this page should be replaced by φ. Line 16 up, *for* $e^{+r\sqrt{-1}}$ and $e^{-r\sqrt{-1}}$ *read* $e^{+v\sqrt{-1}}$ and $e^{-v\sqrt{-1}}$.

Page 45, line 1; *for* 1902 *read* 1903. Footnote (1), *for* 1902 *read* 1903.

Page 83, line 8 up; G. Eneström has pointed out that Euler gave a value of i^i as early as 1746, in a letter to Goldbach. The letter is found in P. H. Fuss, *Correspondance mathématique et physique de quelques célèbres géomètres du XVIII*[ème] *siècle*, tome I, 1843, p. 383. The passage is as follows: *Letztens habe gefunden, dass diese expressio* $(\sqrt{-1})^{\sqrt{-1}}$ *einem valorem realem habe, welcher in fractionibus decimalibus* = 0,2078795763, *welches mir merkwürdig zu seyn scheinet.* Euler makes no reference here to the infinitely many values of i^i.

Page 149, line 3 up; *for* w plane *read* z plane. Line 2 up; *for* values of w *read* values of z.

Page 150, line 6; *for* conection *read* connection. Line 15; *for* z-plane *read* w-plane, *for* w-surface *read* z-surface.

Page 179, lines 15–19 were intended as part of footnote (2).

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[1] *Théorie analyt. de log. népérien et de la fonct. exponent.*, par M. Ch. Méray, Paris, 1891.

THE SIGNIFICANCE OF WEIERSTRASS'S THEOREM.

By E. R. HEDRICK, University of Missouri.

1. Introduction. Certain theorems that are commonly presented in mathematical instruction appear at first sight to be of theoretical importance only. Actual application of some such theorems seems very far fetched, and it is often not sought for because the possibility of its realization seems slight.

Among such theorems is that of Weierstrass on polynomial approximations to a continuous function:*

Given a function $f(x)$ continuous in the interval $a \gtreqless x \gtreqless b$, and a positive number ϵ, then a polynomial $P(x)$ exists, such that

$$|f(x) - P(x)| < \epsilon, \qquad a \gtreqless x \gtreqless b.$$

It is the purpose of this paper to show how this theorem can be clarified to students, and how the contrast between the approximations that it expresses and those given by Taylor's series can be illuminated.

2. Contrast with Taylor Approximating Polynomials. The characteristic property of the successive polynomial approximations

$$(1) \qquad T_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

that are given by a Taylor series, is that *the coefficient a_i of a given power x^i is the same for every approximation past the first in which it occurs.*

The polynomials $P(x)$ of Weierstrass's theorem do not in general have this property, even when $f(x)$ is analytic. If $f(x)$ is not analytic, it is impossible that they should have that property.

To illustrate the manner in which polynomial approximations may arise that do not have the Taylor property, let us consider, for example, the amount of stretching in a steel wire due to variable tension. Hooke's law is expressed by the formula

$$(2) \qquad y = y_0(1 + kx),$$

where y_0 is the original length of the wire, and y is its length under a tension x. The constant k is determined by experiment upon a given wire.

It is known that Hooke's law is only approximately correct. To take into account minor variation not expressed by that law, we may assume a formula of the type

$$(3) \qquad y = y_0(1 + k_1x + k_2x^2),$$

and determine the constants k_1 and k_2 by a series of experiments. If this be done, *the coefficient k_1 of x in* (3) *may or may not, logically, coincide with the value of k in* (2). If it does, it manifests the characteristic Taylor property; if not, the polynomial approximations (2) and (3) are typical rather of the Weierstrass behavior. As a matter of fact, as we shall see below, k_1 is *not* equal to k.

* See, for example, Goursat-Hedrick, *Mathematical Analysis*, Vol. I, p. 422.

For still greater accuracy, or for theoretical purposes, we might insert still higher powers of x in the assumed formula for y:

$$y = y_0(1 + k_1^{(n)}x + k_2^{(n)}x^2 + \cdots + k_n^{(n)}x^n), \tag{4}$$

where $k_i^{(n)}$ denotes the coefficient of x^i in the nth one of these approximations. The question that arises is whether $k_i^{(n)}$ is or is not independent of n; if it is, the polynomials have the Taylor property mentioned above; if not, the coefficient of the same power of x varies from one approximation to another in the Weierstrass manner. We proceed to show that the latter alternative is the usual one.

3. Computation of Coefficients by Least Squares. In such instances as that mentioned in article 2, the coefficients k_i are usually computed by the method of least squares. If the relation between two observed quantities x and y is of the form (4), it is usual to select the values of the coefficients $k_i^{(n)}$ which render the sum of the squares of the differences between the two sides of (4) a minimum, the sum being extended over all the sets of observed values. This is equivalent to solving for $k_i^{(n)}$ the following set of linear equations:

$$\left\{\begin{array}{l}
k_1^{(n)}\Sigma x_j^2 + k_2^{(n)}\Sigma x_j^3 + \cdots + k_n^{(n)}\Sigma x_j^{n+1} = \Sigma x_j\left(\dfrac{y_j - y_0}{y_0}\right), \\
k_1^{(n)}\Sigma x_j^3 + k_2^{(n)}\Sigma x_j^4 + \cdots + k_n^{(n)}\Sigma x_j^{n+2} = \Sigma x_j^2\left(\dfrac{y_j - y_0}{y_0}\right), \\
\quad \cdot \qquad\qquad \cdot \qquad\qquad\qquad \cdot \qquad\qquad \cdot \\
\quad \cdot \qquad\qquad \cdot \qquad\qquad\qquad \cdot \qquad\qquad \cdot \\
\quad \cdot \qquad\qquad \cdot \qquad\qquad\qquad \cdot \qquad\qquad \cdot \\
k_1^{(n)}\Sigma x_j^{n+1} + k_2^{(n)}\Sigma x_j^{n+2} + \cdots + k_n^{(n)}\Sigma x_j^{2n} = \Sigma x_j^n\left(\dfrac{y_j - y_0}{y_0}\right),
\end{array}\right. \tag{5}$$

where x_j and y_j are a pair of corresponding observed values of x and y, and where the summations are extended over all the pairs of observed values.

It is evident from these formulas that the approximation of next higher degree will give a set of coefficients $k_i^{(n+1)}$ that are in general very different from $k_i^{(n)}$, since the inclusion of a term in x^{n+1} in (4) results in increasing the number of equations in (5), as well as in adding a term to each left-hand side. It follows that *the successive approximations given by the theory of least squares have the characteristic Weierstrass property, i. e., the coefficient $k_i^{(n)}$ of x^i depends on n.*

4. Failure of Taylor Series. It is well known that many continuous functions, and even many functions that possess all their derivatives, are not expansible in Taylor series. Thus the function e^{-1/x^2} possesses all its derivatives for every value of x; nevertheless it is not expansible in Taylor series. A function that fails to have a derivative—of no matter how high an order—for a given value of x, is not expansible in Taylor series about that point. It is a result of Weierstrass's theorem that such approximations as (4) above, are not affected by these failures of Taylor's series.

We have seen in Art. 3 that the approximations of the form (4) ought not to be considered as the first terms of a Taylor series. It now becomes evident

that it is not even desirable to do so if it were allowable. For there are serious limitations upon the applicability of Taylor's (infinite) series, whereas *any continuous function* may be treated by Weierstrass's theorem. Doubtless most scientific phenomena are continuous in the range of experimentation; they certainly are so to within the errors of observation.

It follows that the frequent occurence in science of polynomial approximations of the type (4) constitutes an illustration of Weierstrass's theorem rather than of Taylor's theorem. Quite contrary to the spirit of apologetic hesitation that is sometimes manifested by experimenters in using what are popularly known as "the first terms of a Taylor series" to represent an observed quantity, Weierstrass's theorem constitutes a complete justification for the use of formulas of the type (4) in any scientific experiment.

ON THE IMPOSSIBILITY OF CERTAIN DIOPHANTINE EQUATIONS AND SYSTEMS OF EQUATIONS.*

By R. D. CARMICHAEL, Indiana University.

Introduction.

The purpose of this paper is to give simple proofs of several theorems concerning Diophantine equations of specified form and to derive certain interesting consequences of them. All of these proofs are supposed to be novel, in whole or in part; and the results in VI, VII and VIII are believed to be new.

In §1 it is shown that the Diophantine system $q^2 + n^2 = m^2$, $m^2 + n^2 = p^2$ is without a solution (in positive integers). Several of the remaining theorems of the paper are proved by means of this one. In §2 it is shown that no numerical right triangle has a square area. In §3 the impossibility (in positive integers) of the equation $p^4 - q^4 = \alpha^2$ is proved, and several consequences of this fact are obtained. In §4 I show that the equations $m^4 - 4n^4 = \pm t^2$ are both impossible (in positive integers), and derive various consequences.

Incidentally, several proofs are given of the impossibility of the Fermat equation $x^4 + y^4 = z^4$.

The principal arguments in §§1 and 4 are made by means of Fermat's famous method of "infinite descent."

§1. *Impossibility of the simultaneous equations* $q^2 + n^2 = m^2$, $m^2 + n^2 = p^2$.

In this section we shall prove the following theorem:

Theorem I. *There do not exist integers m, n, p, q, all different from zero, such that*

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* Presented to the American Mathematical Society, March 21, 1913.

$$q^2 + n^2 = m^2, \quad m^2 + n^2 = p^2. \tag{1}$$

It is obvious that an equivalent theorem is the following:

THEOREM II. *There do not exist integers m, n, p, q, all different from zero, such that*

$$p^2 + q^2 = 2m^2, \quad p^2 - q^2 = 2n^2. \tag{2}$$

The method of proof is to assume the existence of integers satisfying equations (1) and (2) and to show that we are thus led to a contradiction. The argument we give is an illustration of Fermat's famous method of "infinite descent."

If any two of the numbers p, q, m, n have a common prime factor t, it follows at once from (1) and (2) that all four of them have this factor. For, consider an equation in (1) or in (2) in which these two numbers occur; this equation contains a third number, and it is readily seen that this third number is divisible by t. Then from one of the equations containing the fourth number it follows that this fourth number is divisible by t. Now, let us divide each equation of system (2) by t^2; the resulting system is of the same form as (2). If any two numbers in this resulting system have a common prime factor t_1, we may divide through by t_1^2; and so on. Hence if a pair of simultaneous equations (2) exists then there exists a pair of equations of the same form in which no two of the numbers m, n, p, q have a common factor other than unity. Let this system of equations be

$$p_1^2 + q_1^2 = 2m_1^2, \quad p_1^2 - q_1^2 = 2n_1^2. \tag{3}$$

We shall show that the assumption of a pair of equations (3) leads to a contradiction. Obviously, we may without loss of generality take p_1, q_1, m_1, n_1 to be all positive; and this we do.

From the first equation in (3) it follows that p_1 and q_1 are both even or both odd; and, since they are relatively prime, it follows that they are both odd. Evidently, $p_1 > q_1$. Then we may write

$$p_1 = q_1 + 2\alpha,$$

where α is a positive integer. If we substitute this value of p_1 in the first equation of (3), the result may readily be put in the form

$$(q_1 + \alpha)^2 + \alpha^2 = m_1^2. \tag{4}$$

Since q_1 and m_1 have no common prime factor it is easy to see from this equation that α is prime to both q_1 and m_1, and hence that no two of the numbers $q_1 + \alpha$, α, m_1 have a common factor.

Now it is well known that if a, b, c are positive integers no two of which have a common prime factor, while

$$a^2 + b^2 = c^2,$$

then there exist relatively prime integers r and s, $r > s$, such that

$$c = r^2 + s^2, \quad a = 2rs, \quad b = r^2 - s^2$$

or

$$c = r^2 + s^2, \quad a = r^2 - s^2, \quad b = 2rs.$$

Hence from (4) we see that we may write

$$q_1 + \alpha = 2rs, \quad \alpha = r^2 - s^2 \tag{5}$$

or

$$q_1 + \alpha = r^2 - s^2, \quad \alpha = 2rs. \tag{6}$$

In either case we have

$$p_1^2 - q_1^2 = (p_1 - q_1)(p_1 + q_1) = 2\alpha \cdot 2(q_1 + \alpha) = 8rs(r^2 - s^2).$$

If we substitute in the second equation of (3) and divide by 2 we have

$$4rs(r^2 - s^2) = n_1^2.$$

From this equation and the fact that r and s are relatively prime it follows at once that r, s, $r^2 - s^2$ are all square numbers; say,

$$r = u^2, \quad s = v^2, \quad r^2 - s^2 = w^2.$$

Now $r - s$ and $r + s$ can have no common factor other than 1 or 2; hence from

$$w^2 = r^2 - s^2 = (r - s)(r + s) = (u^2 - v^2)(u^2 + v^2)$$

we see that either

$$u^2 + v^2 = 2w_1^2, \quad u^2 - v^2 = 2w_2^2 \tag{7}$$

or

$$u^2 + v^2 = w_1^2, \quad u^2 - v^2 = w_2^2.$$

And if it is the latter case which arises, then

$$w_1^2 + w_2^2 = 2u^2, \quad w_1^2 - w_2^2 = 2v^2. \tag{8}$$

Hence, assuming equations of the form (2) we are led either to equations (7) or to equations (8); that is, we are led to new equations of the form with which we started. Let us write the equations thus:

$$p_2^2 + q_2^2 = 2m_2^2, \quad p_2^2 - q_2^2 = 2n_2^2; \tag{9}$$

that is, system (9) is identical with that one of systems (7), (8) which actually arises.

Now from (5) and (6) and the relations $p_1 = q_1 + 2\alpha$, $r > s$, we see that

$$p_1 = 2rs + r^2 - s^2 > 2s^2 + r^2 - s^2 = r^2 + s^2 = u^4 + v^4.$$

Hence $u < p_1$. Also,

$$w_1^2 \overline{\overline{<}} w^2 \overline{\overline{<}} r + s < r^2 + s^2.$$

Hence $w_1 < p_1$. Since u and w_1 are both less than p_1 it follows that p_2 is less than p_1. Hence, obviously, $p_2 < | p |$. Moreover, it is clear that all the numbers p_2, q_2, m_2, n_2 are different from zero.

From these results we have the following conclusion: If we assume a system

of the form (2) we are led to a new system (9) of the same form; and in the new system $| p_2 |$ is less than $| p |$.

Now if we start with (9) and carry out a similar argument we shall be led to a new system

$$p_3^2 + q_3^2 = 2m_3^2, \quad p_3^2 - q_3^2 = 2n_3^2,$$

with the relation $| p_3 | < | p_2 |$; and so on *ad infinitum*. But, since there is only a finite number of positive integers less than the given integer $| p |$ this is impossible. We are thus led to a contradiction; whence we conclude at once the truth of II, and likewise of I.

§2. *The area of a numerical right triangle is never a square number.*

This famous theorem of Fermat is readily proved by means of the results in the preceding section. Let the sides and hypotenuse of a numerical right triangle be u, v and w, respectively. The area of this triangle is $\frac{1}{2}uv$. If we assume this to be a square number t^2 we should have the following simultaneous Diophantine equations

$$u^2 + v^2 = w^2, \quad uv = 2t^2. \tag{10}$$

We shall prove our theorem by showing that the assumption of such a system leads to a contradiction.

If any two of the numbers u, v, w have a common prime factor p then the remaining one also has this factor, as one sees readily from the first equation in (10). From the second equation in (10) it follows that t also has the same factor. Then if we put $u = pu_1$, $v = pv_1$, $w = pw_1$, $t = pt_1$, we have

$$u_1^2 + v_1^2 = w_1^2, \quad u_1v_1 = 2t_1^2,$$

a system of the same form as (10). It is clear that we may start with this new system and proceed in the same manner as before, and so on, until we arrive at a system

$$\bar{u}^2 + \bar{v}^2 = \bar{w}^2, \quad \bar{u}\bar{v} = 2\bar{t}^2, \tag{11}$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ are prime each to each.

It is well known that the general solution of the first equation (11) may be written in one of the forms

$$\bar{u} = 2ab, \qquad \bar{v} = a^2 - b^2, \quad \bar{w} = a^2 + b^2;$$

$$\bar{u} = a^2 - b^2, \quad \bar{v} = 2ab, \qquad \bar{w} = a^2 + b^2.$$

Then from the second equation in (11) we have

$$\bar{t}^2 = ab(a^2 - b^2) = ab(a - b)(a + b).$$

It is easy to see that no two of the numbers a, b, $a - b$, $a + b$ in the last member of this equation have a common factor; for, if so, $\bar{u}$ and $\bar{v}$ would have a common factor, contrary to hypothesis. Hence each of these four numbers is a square. That is, we have equations of the form

$$a = m^2, \quad b = n^2, \quad a + b = p^2, \quad a - b = q^2;$$

whence

$$m^2 - n^2 = q^2, \quad m^2 + n^2 = p^2.$$

But, according to theorem I, no such system of equations can exist.

Hence we have the following theorem:

THEOREM III. *The area of a numerical right triangle is never a square number.*

COROLLARY. *The equation $\alpha^4 + 4\beta^4 = \gamma^2$ is impossible in integers α, β, γ, all of which are different from zero.*

For, from $\alpha^4 + 4\beta^4 = \gamma^2$ it would follow that a right triangle whose sides and hypotenuse are respectively α^2, $2\beta^2$, γ has the square area $\alpha^2\beta^2$, which is impossible.

In the next section additional proofs will be given of this corollary and theorem.

§3. *Impossibility of the equation $p^4 - q^4 = \alpha^2$. Consequences.*

If there exists a set of integers p, q, α all different from zero, such that

$$p^4 - q^4 = \alpha^2, \tag{12}$$

it is easy to see that there exists such a set having also the further property that they are prime each to each. For, if any two of the numbers contain the prime factor t, the third contains this factor also and the equation may be divided through by t^4. Continuing this process one arrives finally at an equation of the form (12) in which p, q, α are prime each to each.

We shall now show that the assumption of such an equation (12) leads to a contradiction of theorem I or of theorem II. We have

$$(p^2 - q^2)(p^2 + q^2) = \alpha^2.$$

Since p and q are relatively prime the two factors of the first member of this equation evidently have the greatest common factor 1 or 2; for the greatest common factor is a divisor of $2q^2$, the difference of the numbers. Hence $p^2 - q^2$ and $p^2 + q^2$ are either both square numbers or both the double of square numbers. That is, we have a system of the form

$$p^2 - q^2 = m^2, \quad p^2 + q^2 = n^2,$$

or a system of the form

$$p^2 - q^2 = 2m^2, \quad p^2 + q^2 = 2n^2.$$

But, according to theorems I and II, each of these systems is impossible.

Thus we have the following theorem:

THEOREM IV. *There do not exist integers p, q, α, all different from zero, such that*

$$p^4 - q^4 = \alpha^2;$$

that is, the difference of two fourth power integers cannot be a square.

Second proof. This theorem may also be readily proved by means of theorem III. For, assuming (12) we have

$$(p^4 - q^4)p^2q^2 = p^2q^2\alpha^2. \tag{13}$$

But, obviously,

$$(14) \qquad (2p^2q^2)^2 + (p^4 - q^4)^2 = (p^4 + q^4)^2.$$

Now, from (13) we see that the numerical right triangle determined by (14) has its area $p^2q^2(p^4 - q^4)$ equal to the square number $p^2q^2\alpha^2$. But this is impossible. Hence no equation of the form (12) exists.

COROLLARY. *The system* $p^2 - q^2 = km^2$, $p^2 + q^2 = kn^2$ *is impossible in integers* p, q, k, m, n, *all of which are different from zero.*

For, if we take the product of these two equations member by member we have as a result an equation which by theorem IV is impossible.

Since from $x^4 + y^4 = z^4$, we have $z^4 - x^4 = (y^2)^2$ the above theorem leads to the following:

THEOREM V. *The equation* $x^4 + y^4 = z^4$ *has no solution in integers all of which are different from zero.*

By means of theorem IV one may readily prove the corollary to theorem III. Let us assume an equation of the form

$$\alpha^4 + 4\beta^4 = \gamma^2.$$

If α and β have a common prime factor p, γ has also this factor, and we may divide the equation by p^4, the resulting equation being of the same form as the original one. This process may be repeated until we have an equation

$$\alpha_1{}^4 + 4\beta_1{}^4 = \gamma_1{}^2,$$

where α_1 and β_1 are relatively prime. It is then easy to see that γ_1 has no odd factor in common with either α_1 or β_1. Suppose that α_1 and γ_1 have the factor 2 in common. We may then divide the equation by 4, thus obtaining a new equation of the same form. We may continue this process until the α and the γ of the resulting equation are not both even. It is then obvious that both must be odd. Let this equation be

$$\bar{\alpha}^4 + 4\bar{\beta}^4 = \bar{\gamma}^2.$$

Then $\bar{\alpha}^2$, $2\bar{\beta}^2$ and $\bar{\gamma}$ are relatively prime. Then from the theory of numerical right triangles it follows that we have equations of the form

$$\bar{\beta}^2 = ab, \qquad \bar{\alpha}^2 = a^2 - b^2, \qquad \bar{\gamma} = a^2 + b^2;$$

where a and b are relatively prime positive integers. From the first of these equations we see that a and b are both square numbers. Then the second of the equations is of the form (12). Since no equation of this form exists we conclude the impossibility of the equation $\alpha^4 + 4\beta^4 = \gamma^2$; that is, we have the corollary of theorem III.

Having demonstrated this corollary without the use of theorem III we may now employ the corollary to prove III itself. Let the argument read as in the preceding case down to equations (11). From the second equation in (11) we see that $\bar{u}$ and $\bar{v}$ have this property: One of them is a square number, say α^2,

and the other is twice a square number, say $2\beta^2$. Then the first equation in (11) takes the form $\alpha^4 + 4\beta^4 = \bar{w}^2$. This equation being impossible we conclude the truth of theorem III.

§4. *Impossibility of the equations* $m^4 - 4n^4 = \pm t^2$. *Consequences.*

We shall assume the existence of one of the equations

$$m^4 - 4n^4 = \pm t^2, \tag{15}$$

in which m, n, t are all different from zero, and show that we are led to a contradiction. If any two of the numbers m, n, t have a common odd prime factor p then all three of them have this factor, and the equation may be divided through by p^4. The new equation thus obtained is of the same form as the original one. This process may be repeated until an equation

$$m_1^4 - 4n_1^4 = \pm t_1^2$$

is obtained in which no two of the numbers m_1, n_1, t_1 have a common odd prime factor.

If t_1 is even it is obvious that m_1 is also even, and therefore the above equation may be divided by 4; a result of the form

$$n_1^4 - 4m_2^4 = \mp t_2^2$$

is obtained. The process can be continued until an equation

$$m_3^4 - 4n_3^4 = \pm t_3^2$$

is obtained in which t_3 is an odd number. Then m_3 is also odd. If the second member of the above equation has the minus sign then we have

$$(m_3^2)^2 + t_3^2 = 4n_3^4;$$

that is, we have the sum of two odd squares divisible by 4, which is impossible since an odd square is congruent to 1 modulo 4. Hence we must have

$$4n_3^4 + t_3^2 = m_3^4. \tag{16}$$

Moreover, it is obvious that no two of the numbers n_3, t_3, m_3 have a common prime factor and that all of them are different from zero.

From (16) and the theory of numerical right triangles we have equations of the following form:

$$n_3^2 = rs, \qquad t_3 = r^2 - s^2, \qquad m_3^2 = r^2 + s^2,$$

where r and s are relatively prime positive integers. From the first of these equations it follows that r and s are squares; say, $r = \rho^2$, $s = \sigma^2$. Then from the last equation we have

$$\rho^4 + \sigma^4 = m_3^2.$$

It is easy to see that ρ, σ and m_3 are prime each to each.

For this equation we have the general solution

$$m_3 = r_1^2 + s_1^2, \quad \rho^2 = 2r_1s_1, \quad \sigma^2 = r_1^2 - s_1^2,$$

or the general solution

$$m_3 = r_1^2 + s_1^2, \qquad \rho^2 = r_1^2 - s_1^2, \qquad \sigma^2 = 2r_1s_1.$$

In either case we see that $2r_1s_1$ and $r_1^2 - s_1^2$ are both squares, while r_1 and s_1 are relatively prime and one of them is even. From the relation $r_1^2 - s_1^2 =$ square, it follows that r_1 is odd, since otherwise we should have the sum of two odd squares equal to the even square r_1^2, which is impossible. Hence s_1 is even. But $2r_1s_1 =$ square. Then, since r_1 is odd we have r_1 equal to a square. Then $2s_1$ is a square, or s_1 is twice a square. Then we may write $r_1 = \rho_1^2$, $s_1 = 2\sigma_1^2$. Hence we have an equation of the form

$$\rho_1^4 - 4\sigma_1^4 = w_1^2$$

since $r_1^2 - s_1^2$ is a square; that is, we have

$$4\sigma_1^4 + w_1^2 = \rho_1^4. \tag{17}$$

Now the last equation has been obtained solely from (16). Moreover it is obvious that no one of the numbers σ_1, w_1, ρ_1 is zero. Also, we have

$$n_3^2 = rs = \rho^2\sigma^2 - 2r_1s_1(r_1^2 - s_1^2) = 4\rho_1^2\sigma_1^2(\rho_1^4 - 4\sigma_1^4) = 4\rho_1^2\sigma_1^2w_1^2.$$

Hence $\sigma_1 < n_3$. Similarly, starting from (17) we should be led to an equation

$$4\sigma_2^4 + w_2^2 = \rho_2^4,$$

where $\sigma_2 < \sigma_1$; and so on indefinitely. But such a recursion is impossible.

We have been led to this contradiction by the initial assumption that an equation of the form (15) is possible. Hence we have the following theorem:

THEOREM VI. *Neither of the equations $m^4 - 4n^4 = \pm t^2$ is possible in integers m, n, and t, all of which are different from zero.*

COROLLARY. *The system of equations $p^2 - 2q^2 = km^2$, $p^2 + 2q^2 = \pm kn^2$ is not possible in integers p, q, k, m, n, all of which are different from zero.*

For, if we take the product of these equations member by member we have as a result an equation which by theorem VI is impossible.

From the corollary to theorem VI we have also the following interesting result:

THEOREM VII. *The area of a numerical right triangle is not twice a square number.*

For, if there exists a set of numbers u, v, w such that

$$u^2 + v^2 = w^2, \qquad uv = 4k^2 = t^2$$

then it is easy to see that

$$(u + v)^2 = w^2 + 2t^2,$$

$$(u - v)^2 = w^2 - 2t^2.$$

Since from the preceding corollary these equations are impossible, the truth of the theorem follows at once.

From theorems III and VII we have the following corollary:

COROLLARY. *The number expressing the area of a numerical right triangle has at least one odd prime factor entering to an odd power.*

Now a number of the form $rs(r^2 - s^2)$, where r and s are different positive integers, is the area of the numerical right triangle determined by the equation

$$(2rs)^2 + (r^2 - s^2)^2 = (r^2 + s^2)^2.$$

Put $r = \rho^2$ and $s = \sigma^2$. Then from the above corollary we see that $\rho^2\sigma^2(\rho^4 - \sigma^4)$ has some odd prime factor entering into it to an odd power. Since every prime factor of $\rho^2\sigma^2$ obviously enters into $\rho^2\sigma^2$ to an even power, we have the following result:

THEOREM VIII. *The number $\rho^4 - \sigma^4$, in which ρ and σ are different positive integers, has always an odd prime factor entering into it to an odd power.*

As immediate corollaries of this theorem we have theorems IV and V in §3.

By means of theorem VII we may also prove the following theorem:

THEOREM IX. *The equation $m^4 + n^4 = \alpha^2$ is impossible in integers all of which are different from zero.*

For, if such an equation exists we have a right triangle $(m^2)^2 + (n^2)^2 = \alpha^2$ whose area $\frac{1}{2}m^2n^2$ is twice a square number, which is impossible.

Another proof of theorem V by means of theorem IX is immediate. Theorems IV and IX, taken together, may be stated in the form:

THEOREM X. *In a numerical right triangle $a^2 + b^2 = c^2$, not more than one of the numbers a, b, c is a square.*

PROBLEMS AND QUESTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

ALGEBRA.

392. Proposed by H. PRIME, Boston, Mass.

The floor and ceiling of a room h feet high are parallel and horizontal. In the middle of the ceiling is a vertical circular tube w feet in diameter extending upward from the ceiling. From the room the longest possible inflexible rod is put up the tube. When the rod is in contact with the opposite sides of the tube and the floor, what is the expression for the tangent of the acute angle made by the rod and the floor in terms of h and w, considering the rod as an air line. To be solved without the calculus. [From *The Maine Farmers' Almanac*, 1913.]

393. Proposed by H. E. TREFETHEN, Waterville, Maine.

Expand $\log [(\sin x)/x]$ in terms of x.

394. Proposed by E. B. ESCOTT, University of Michigan.

Solve the equation $\sin^2 x \sin^2 2x = 5/16$.

395. Proposed by V. M. SPUNAR, Chicago, Ill.

Solve the system of equations $x_1^2x_2 = a_1$; $x_2^2x_3 = a_2$; $x_3^2x_4 = a_3$; $\cdots$ $x_n^2x_1 = a_n$.

GEOMETRY.

420. Proposed by C. N. SCHMALL, New York City.

Four spheres are described so that each touches a face of a given triangular pyramid and the other three faces produced. If the radii of the escribed spheres be r_1, r_2, r_3, r_4, and r be the radius of the inscribed sphere, show that

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}=\frac{2}{r}.$$

421. Proposed by R. P. BUSSEY, University of Iowa.

Assuming the details of the proof of the existence of a sphere inscribed in a tetrahedron as usual in the texts, give an intuitional proof that there are in general eight spheres each touching the four faces, but for the regular tetrahedron only five. How many special types are there?

422. Proposed by J. SCHEFFER, Hagerstown, Md.

Construct strictly by use of elementary plane geometry, a triangle having given the product of two sides, the median to the third side, and the difference of the angles adjacent to the third side.

423. Proposed by C. N. SCHMALL, New York City.

The sum of the squares of the distances of a point from n fixed points is constant. Show that the locus of the point is a circle.

CALCULUS.

342. Proposed by CLARIBEL KENDALL, University of Colorado.

Referring to Poincaré's *Science and Hypothesis*, page 65, show that the path between two points, in Poincaré's ideal world, requiring the least number of steps of beings such as exist on earth, is the arc of a circle cutting the boundary of the Poincaré world orthogonally.

343. Proposed by C. N. SCHMALL, New York City.

Show that the envelope of the system of circles, whose radii are the ordinates of an ellipse, is a concentric ellipse having the same minor axis as the given ellipse. Does this ellipse touch all the circles of the system?

344. Proposed by B. F. FINKEL, Drury College.

Solve the differential equation,

$$\frac{d^2y}{dx^2}(a^2-x^2-y^2)=\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}.$$

337. *Correction.* For $\int_0^1(\cos 2\pi ax+\cos 2\pi bx)dx$ read $\int_0^1 \mid \cos 2\pi ax+\cos 2\pi bx \mid dx$.

MECHANICS.

273. Proposed by A. M. HARDING, University of Arkansas.

A spherical shell of mass m explodes when moving with negligible velocity at a height of h feet above the ground. The shell is divided into very small particles, each of which moves, after the explosion, away from the center of the shell with a velocity v, and ultimately falls to the ground. Find the total mass of the fragments which will be found per unit area at any specified distance from the point vertically underneath the shell.

274. Proposed by W. W. LANDIS, Dickinson College.

A dam backs up the water for two miles. If the dam is raised 18 inches, will the water two miles up the stream be raised 18 inches, more or less?

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

190. Proposed by L. E. DICKSON, University of Chicago.

Find two numbers a and b each of two digits such that, when the product ab is found by the usual method, the two partial products to be added are exactly the same as the partial products in getting the product ba. Is there a similar pair of numbers of three digits?

191. Proposed by E. B. ESCOTT, University of Michigan.

Find triangles whose sides are integers and one of whose angles is 60°.

192. Proposed by CHARLES MACAULEY, Chicago, Ill.

Combinations containing an even number of letters are formed of the letters a, b, c, d, etc. It is required to place the letters in two columns, so that half the letters in every combination are placed in one column and the other letters of the combination in the other column, and so that all the a's are placed in the same column; all the b's in the same column; all the c's in the same column, etc.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

383. Proposed by E. B. ESCOTT, University of Michigan.

Find accurately to 6 decimals $1.000000854^{1283451}$.

SOLUTION BY THE PROPOSER.

Using natural logarithms,

$$\log_e (1 + x) = x - \frac{x^2}{2} + \cdots.$$

Then

$$\log_e 1.000000854 = .000000854 -, + \cdots,$$

and the natural logarithm of

$$1.000000854^{1283451} \text{ is } 1.096066686.$$

Hence the common logarithm is $1.096066686 \times .434294482$, or .4760157. And the number required is 2.992373.

384. Proposed by H. C. FEEMSTER, York, Nebraska.

A man addressed n envelopes and wrote n checks in payment of n bills. Show that the number of ways of enclosing within each envelope one bill and one check in such a manner that in no instance all the enclosures shall be correct is

$$n!\left\{n! - \frac{(n-1)!}{1!} + \frac{(n-2)!}{2!} \cdots + (-1)^n \frac{0!}{n!}\right\}, \text{ where } 0! = 1.$$

SOLUTION BY THE PROPOSER.

The number of ways in which the enclosures in one letter may be correct is ${}_nC_1((n-1)!)^2$ except ${}_nC_2((n-2)!)^2$ which have been counted twice, except also ${}_nC_3((n-3)!)^2$ of these latter, which have been counted twice, etc. So the number of ways in which the enclosures in one letter may be correct is

$$ {}_nC_1((n-1)!)^2 - {}_nC_2((n-2)!)^2 + {}_nC_3((n-3)!)^2 - \cdots + (-1)^{n-1}{}_nC_{n-2}(2!)^2 + (-1)^n\, {}_nC_{n-1}(1!)^2 + (-1)^{n+1}\, {}_nC_n(0!)^2.$$

Hence the number of ways in which both enclosures in a single envelope may be incorrect is:

$$(n!)^2 - {}_nC_1((n-1)!)^2 + {}_nC_2((n-2)!)^2 - {}_nC_3((n-3)!)^2 + \cdots + (-1)^{n-2}\, {}_nC_{n-2}(2!)^2 + (-1)^{n-1}{}_nC_{n-1}(1!)^2 + (-1)^n{}_nC_n\, 0!'$$

or

$$(n!)^2 - \frac{n((n-1)!)^2}{1!} + \frac{n(n-1)}{2!}((n-2)!)^2 - \frac{n(n-1)(n-2)}{3!}((n-3)!)^2$$

$$+ \cdots (-1)^{n-2}\frac{n(n-1)}{2!}(2!)^2 + (-1)^{n-1}n(1!)^2 + \cdots + (-1)^n 1\cdot 0!,$$

where $0!$ is defined as 1. Or it may be written:

$$n!\left[n! - \frac{(n-1)!}{1!} + \frac{(n-2)!}{2!} - \frac{(n-3)!}{3!} + \cdots + (-1)^{n-2}\frac{2!}{(n-2)!} + (-1)\frac{1!}{(n-1)!} + (-1)^n\frac{0!}{n!}\right].$$

GEOMETRY.

412. Proposed by S. LEFSCHETZ, University of Nebraska.

To inscribe in a given circle an isosceles triangle in which the base plus the altitude is equal to a given length.

SOLUTION BY A. M. HARDING, Univ. of Arkansas.

From A, any point on the circle, draw a diameter and produce it to P making $AP = l$, the given length. At A draw AQ tangent to the circle and equal to $\frac{1}{2}l$. Join P and Q. Assume that the line PQ cuts the circle in two points B and B'.

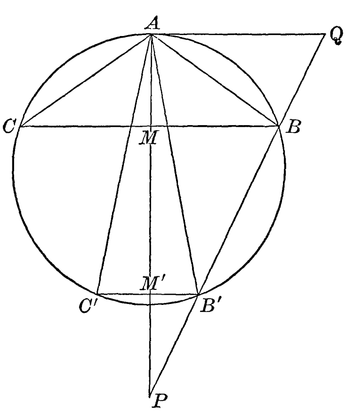

Fig. 1.

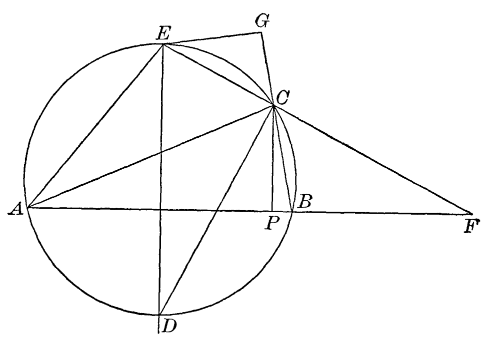

Fig. 2.

From B and B' draw chords perpendicular to AP cutting AP at M and M' and cutting the circle at C and C' respectively. Then ABC and $AB'C'$ are the isosceles triangles required. See Fig. 1.

Proof.

$$\frac{PM}{MB} = \frac{PA}{AQ} = \frac{2}{1}.$$

Hence

$$PM = 2MB = CB.$$

Then $AM + CB = AM + PM = l$. Likewise, $AM' + C'B' = l$. There will be two solutions, one solution, or no solution according as PQ cuts the circle, is tangent to the circle, or lies outside of the circle. In other words the problem is possible only when $2r < l < (1 + \sqrt{5})r$.

Also solved analytically, by H. C. FEEMSTER.

413. Proposed by DAVID R. KELLEY, New York City.

To construct a triangle, given the base, the vertical angle, and the ratio of the altitude from the vertex on the base to the difference of the other two sides.

SOLUTION BY THE PROPOSER.

Let ABC be the required triangle. Then since side AB and $\angle C$ are known, we can construct the circumcircle of the triangle. Let the bisectors of the interior angle C and the exterior angle C meet the circle in D and E, respectively. Then D and E are fixed points, and ED is the diameter of the circle, and AE is fixed. Draw EG perpendicular to BC produced. Then $CG = \frac{1}{2}(AC - BC)$. Hence, if CP be the altitude from C on AB, CG/CP is also known, since by hypothesis $(AC - CB)/CP$ is the given ratio. Again, since in the right triangle ECG, $\angle GCE$ is known, therefore GC/EC is known, and therefore, by above, EC/CP is known. Now let EC meet AB in F. Then triangles DEC and CFP are similar. Hence, $EC/CP = DE/CF$. But EC/CP is given. Hence DE/CF is also known, and therefore, since DE is the diameter of the circumcircle of triangle ABC, CF can be found. Also $EC \cdot EF = \overline{EA}^2$. Hence, since CF and EA are known, EC can be constructed and hence point C can be found. See Fig. 2.

Also solved by H. C. FEEMSTER, C. N. SCHMALL, and A. M. HARDING.

414. Proposed by H. C. FEEMSTER, York, Neb.

To construct the cyclic quadrilateral, having given its four sides.

SOLUTION BY THE PROPOSER.

Construction.—Let the lengths of the given sides be a, b, c, d. Draw $AD = d$, and produce it to E so that $DE : c = a : b$. Now describe a circle which is the locus of a point P such that $EP : PA = c : b$ (circle of Apollonius). Describe also a circle with the center D and radius c. Let C be one point of intersection of these circles. Then on AC as a base construct a triangle ABC similar to EDC. Then $ABCD$ is the quadrilateral required. See Fig. 3.

Proof.—By construction $AD = d$, $DC = c$. Now since the angles ABC and EDC are equal, therefore $\angle ABC$ and $\angle ADC$ are supplementary and the figure $ABCD$ is cyclic. It now remains to prove that $AB = a$, and $BC = b$.

By construction, $ED : c = a : b$, or $ED : DC = a : b$. Also by construction $EC : CA = c : b$, or $EC : DC = CA : b$. Hence

$$ED : DC : EC = a : b : CA. \tag{1}$$

But by similar triangles, we have

$$ED : DC : EC = AB : BC : CA. \tag{2}$$

Then by (1) and (2), $AB = a$, $BC = b$.

There is evidently but one solution; for if the other point of intersection of the circles be taken instead of C, the resulting quadrilateral will be the same in size but drawn on the opposite side of AD since the intersections are symmetrical with regard to AD.

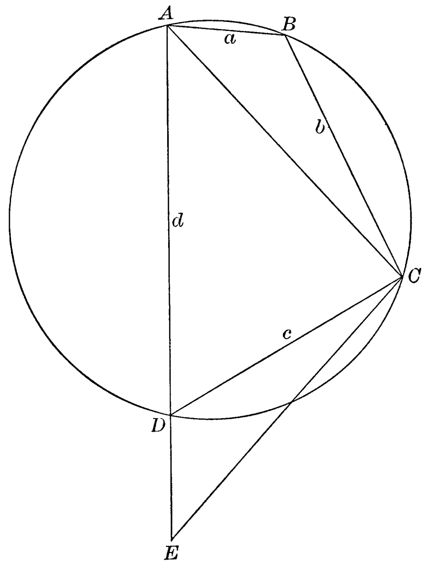

Fig. 3.

Note.—In this construction it is assumed that the order of the sides a, b, c, d, is given. If this order is not given it may be shown that three cyclic quadrilaterals can be drawn having these sides.

MECHANICS.

265. Proposed by A. H. HOLMES, Brunsw'ck, Maine.

A gun is mounted in a fort at height h above the sea, and a similar gun is mounted on a ship. Show that there is a region of area $4\pi rh$ within which the ship is within range of the fort while the fort is out of range of the ship, r being the maximum range of either gun on a horizontal plane through it.

SOLUTION BY A. M. HARDING, University of Arkansas.

If a projectile is fired at an angle α its range R on a level plane at a distance h below the point of projection is given by the equation

$$2v^2(h + R\tan\alpha) = gR^2\sec^2\alpha.$$

That is

$$R = \frac{v}{g}\cos\alpha\,[v\sin\alpha + \sqrt{(v^2 + 2gh) - v^2\cos^2\alpha}].$$

From the value of $dR/d\alpha$, set equal to zero, we have as the condition for a maximum range,

$$\sin^2\alpha = \frac{v^2}{2(v^2 + gh)} \qquad \text{and} \qquad \cos^2\alpha = \frac{v^2 + 2gh}{2(v^2 + gh)}.$$

Hence $R = \sqrt{r^2 + 2hr}$, where r is the maximum horizontal range. This is the maximum range for a gun in the fort. Likewise the maximum range for a gun on the ship is

$$R_1 = \sqrt{r^2 - 2hr}.$$

Therefore the area required is

$$\pi R^2 - \pi R_1^2 = 4\pi hr.$$

Also solved by MRS. H. E. TREFETHEN and H. C. FEEMSTER.

NUMBER THEORY.

185. Proposed by R. D. CARMICHAEL, Bloomington, Ind.

Obtain the complete solution of the equation $\phi(p^\alpha) = \phi(q^\beta)$ where ϕ denotes Euler's ϕ-function, p and q are unknown primes, and α and β are unknown integers.

SOLUTION BY L. C. MATHEWSON.

From the theory of numbers[1] we have

$$\begin{aligned} \phi(p^\alpha) &= p^\alpha\left(1 - \frac{1}{p}\right) = p^{\alpha-1}(p-1); \\ \phi(q^\beta) &= q^\beta\left(1 - \frac{1}{q}\right) = q^{\beta-1}(q-1). \end{aligned} \tag{1}$$

Obviously the given equation is always satisfied by $p = q$, if $\alpha = \beta$. We, accordingly, consider the case where the primes are not equal, or $p \neq q$.

From (1) we have

$$p^{\alpha-1}(p-1) = q^{\beta-1}(q-1). \tag{2}$$

Since $p^{\alpha-1}$ and $q^{\beta-1}$ can have no common factors, we see that (2) gives

$$\begin{aligned} p^{\alpha-1} &= q - 1 \\ q^{\beta-1} &= p - 1 \end{aligned}. \tag{3}$$

It is sufficient to consider only the first of these equations, since the second is of the same form and leads to no new general results or solutions. Now $\alpha = 1$ or $\alpha > 1$. From (3) written in the form $p^{\alpha-1} + 1 = q$, we see that if $\alpha = 1$, $q = 2$, the even prime: if $\alpha > 1$, p cannot be an odd prime (for this would make q composite), hence p can be only 2. Since we are now excluding $p = q$, and because from (3) we have seen that the results will be interchangeable, and since we now see that at least one of the primes here must be the even prime, we consider p odd and q even. We have then to discuss $\phi(p^\alpha) = \phi(q^\beta)$, where p is an odd prime, $\alpha = 1$, $q = 2$; i. e., $\phi(p) = \phi(2^\beta)$.

We have simply $p - 1 = 2^{\beta-1}$, or $p = 2^{\beta-1} + 1$. Here p is an odd prime of the form $2^m + 1$. It is easily seen that m cannot be an odd prime nor an integral multiple of an odd prime (for in these cases since the binomial form itself is the

[1] See Bachmann's Neuere Zahlentheorie, p. 24, or any elementary text on the theory of numbers.

sum of the same odd powers of two integers, it is factorable); m can thus be only an integral power of 2; as, 1, 2, 4, 8, ,$\cdots$. This means that $\beta = 2^n + 1$, where n must be such a positive integer that $p = 2^{2^n} + 1$ is a prime.

Summarizing, we have for solutions of the given equation $\phi(p^\alpha) = \phi(q^\beta)$:

Case (1), $p = q$. Any prime, even or odd, is a root, if $\alpha = \beta$.

Case (2), $p \neq q$. One root is the even prime 2 with any exponent 2^n such that $2^{2^n} + 1$ is a prime, this odd prime being the other root for this individual solution. In this case the exponent of the odd prime is unity, while that of the even prime is of the form 2^n, as just indicated.

It may be added that the general form for n such that $2^{2^n} + 1$ is always a prime is still undetermined. It is of interest to note that this form is one which Fermat conjectured was always a prime for positive integral values of n. While this is true for many values of n, as 0, 1, 2, 3, $\cdots$, the factors have been found for other values of n: thus $2^{2^n} + 1$ is composite for $n = 5, 6, 9, 11, 12, 18, 23, 36, 38$.[1] These numbers are too large to be handled readily (or at all) when expressed in ordinary Arabic notation. The French encyclopedia of mathematics (t. 1: 3, p. 51) states that $2^{2^{36}}$ is an integer of more than twenty billion digits. The two following interesting propositions have been proposed, but no published proofs seem to exist: All the numbers represented in the series $2 + 1$, $2^2 + 1$, $2^{2^2} + 1$, $2^{2^{2^2}} + 1$, $\cdots$ are primes. The second is due to Eisenstein: There is an infinity[2] of prime integers of the form $2^{2^n} + 1$.

186. Proposed by H. PRIME, Boston, Mass.

Show that $\dfrac{(n+1)(n+2)\cdots(2n-2)}{(n-1)!}$ is an integer for all values of n.

SOLUTION BY THOMAS E. MASON, Bloomington, Ind.

The expression in the problem is equivalent to (1) $(2n-2)!/n!(n-1)!$ and the expression (2) $(m+n)!/m!n!$ is an integer, since it is a coefficient in the binomial expansion for the exponent $m + n$.

Each prime factor of the denominator of (1) which is not a factor of n occurs in the denominator as many times as it occurs in $(n-1)!(n-1)!$, and therefore occurs at least as many times in the numerator, since by (2) $(2n-2)!/[(n-1)!(n-1)!]$ is an integer.

Any prime factor of n (different from 1) is not contained in $n - 1$, and therefore occurs in the denominator of (1) as many times as it occurs in $n!(n-2)!$. It occurs at least as many times in the numerator since by (2) $(2n-2)!/n!(n-2)!$ is an integer.

The expression in the problem is therefore an integer for all values of n since the numerator contains every prime factor of the denominator at least as many times as it occurs in the denominator.

Also solved by B. F. YANNEY and H. C. FEEMSTER.

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[1] Intermed. Math., 10 (1903), p. 158; 11 (1904), p. 79. Lucas, Théorie des Nombres, p. 51.

[2] Lucas, Théorie des Nombres, p. 355.

BOOK REVIEWS.

School Algebra. Book I. By George Wentworth and David Eugene Smith. Ginn and Company, Boston, 1913. iv + 298 pages. $0.90.

This is the third of the Wentworth-Smith Series. The authors have prepared a two years course in algebra for high schools, to be issued in two volumes, of which this is the first. The plan of the work is similar to that of their Academic Algebra, reviewed in the June number of the Monthly, but the treatment of topics is somewhat more extended.

R. R. Shumway.

Les Anaglyphes Géométriques. By H. Vuibert. Librairie Vuibert, Paris, 1912. 32 pages. Francs 1.50.

Some sixty years ago Wilhelm Rollmann of Stralsund published a paper on the use of two-colored images with glasses of two colors (not necessarily complementary). In 1874, 1875 and 1876 L. Ducos du Hauron of Algiers published several articles on héliochromie. The latter investigator imposed one upon the other two stereoscopic views in complementary colors and gave to the picture thus formed the name *anaglyphe*. By looking at this picture with the right eye through a red medium, with the left eye through a green, a marvelous stereoscopic effect is secured, the figure seeming to float in space.

Profiting by the discoveries of Ducos du Hauron, M. Richard exhibited at the international congress of mathematicians in Cambridge last year some forty anaglyphs which were much admired. More recently MM. Richard and Vuibert have published this little book of 32 pages containing thirty specimen anaglyphs illustrative of solid and descriptive geometry, crystallography and physics. Each copy contains a pair of red and green "glasses" mounted in card board. It is the purpose of the authors to issue a series of albums of anaglyphs some of which will be of special interest to teachers in secondary schools.

W. W. Beman.

Trigonometry. By Alfred Monroe Kenyon and Louis Ingold. Edited by Earle Raymond Hedrick. The Macmillan Company, New York, 1913. xi + 132 pages. *Logarithmic and Trigonometric Tables.* Prepared under the direction of Earle Raymond Hedrick. The Macmillan Company, New York, 1913. xvii + 124 pages. $1.35.

The Kenyon-Ingold Trigonometry is the second of the new series of mathematical books published by The Macmillan Company under the general editorship of Professor E. R. Hedrick, of the University of Missouri. Like the Davis Calculus, the first of the new series, it departs somewhat from the traditional course. The first marked departure is that the solution of triangles is attacked immediately and is completely solved in the first three chapters (47 pages), the addition formulas, trigonometric equations and identities, and all references to angles greater than 180° being left out because they are extraneous. The trig-

onometric functions are first defined for acute angles and are used in solving right triangles. Then they are defined for obtuse angles and are used in solving oblique triangles. The student is taught the fundamental principles of solution by means of the *law of sines* and the *law of cosines*. The table of powers and roots at the back of the book makes the latter law practicable for computation. Then he is taught the special logarithmic methods that make use of the *law of tangents* and the *law of tangents of the half-angles*. Geometric proofs are given for these two laws. The proofs usually given are not available because they involve the addition formulas and the formulas for double and half angles which come later in this book.

The first sentence of the preface sounds the keynote of the book. "In trigonometry, as elsewhere, a motive for the study of each topic is necessary to secure the effective attention of the student." The motive so far in the book is the solution of triangles. In most of the current texts, this is made to serve as the only motive. The solution of triangles is begun early and finished late in the course. After the student has learned to solve right triangles, he begins the study of what is called goniometry or trigonometric analysis. He learns about the functions of angles of all sizes, and about the addition formulas and the formulas to which they lead. He knows that the goal toward which he is striving is the solution of oblique triangles; and after he has reached that goal, he wonders why he has had to solve so many equations and prove so many formulas and identities that seem to him unnecessary. The solution of triangles is one of the most interesting problems of elementary mathematics, and it will be more interesting and more satisfactory to the student if it is presented in a straightforward manner with everything extraneous left out, as it is done in this book.

The other parts of trigonometry, which are more important for analytic geometry and the calculus, are taken up in chapters 4, 5, 6, and 7. Directed lines and angles, radian measure, functions of any angle, graphs, the addition formulas, equations and identities, and the inverse functions follow in order. The composition and resolution of forces is made an introduction to the study of angles greater than 180° and is used to illustrate the meaning of the addition formulas. Large angles are used in problems on rotation and angular speed, and the radian is shown to be useful in problems on rotation and mensuration. The applications of large angles include forces, velocities, simple harmonic motion, and vibrations.

So far in the book a motive has been found for each topic. There has been a minimum of abstract trigonometric manipulation. The next chapter deals with identities, equations, and inverse functions. It is unusually complete and well written, especially the part on trigonometric equations. It contains no practical applications.

The authors have used rectangular coordinates from the very beginning of the book. Many teachers believe that plotting with polar coordinates should be a part of every course in trigonometry because it gives the student facility in handling the trigonometric functions, especially those of angles greater than 180°,

and because it is a good preparation for analytic geometry. The authors have not put this topic in their book. Perhaps they thought it would have to be dragged in without any motive that the student could see.

The last chapter in the book (25 pages) is devoted to Spherical Trigonometry.

The logarithmic and trigonometric tables prepared under the direction of E. R. Hedrick are unusually complete. They are five-place tables and include tables of powers, roots, and reciprocals, and natural logarithms. Brief four-place tables are added for use in problems that do not demand extreme accuracy. The introduction explains the slide rule and gives a graphic representation of interpolation.

Answers to problems are not given.

W. H. Bussey.

NOTES AND NEWS.

The August number of *The Popular Science Monthly* contains an article entitled "Bernoulli's Principle and its application to explain the curving of a base ball."

Mr. E. J. Moulton has been promoted to an assistant professorship in mathematics at Northwestern University. He has just taken the doctor's degree at the summer convocation at the University of Chicago.

John Wiley and Sons have recently published "The Theory of Relativity," by Professor Robert D. Carmichael, of the University of Indiana, as No. 12 of the series of mathematical monographs edited by Mansfield Merriman and Robert S. Woodward. The price is $1.00.

Mr. H. R. Kingston, graduate student at the University of Chicago, has been appointed to an instructorship in mathematics at the University of Manitoba, Winnipeg, Canada.

The Macmillan Company has published an *edition de luxe* of "The Davis Calculus," printed on India paper and bound in flexible leather. The thickness of the book is one half that of the regular edition. The price is $2.40. The price of the regular edition is $2.00.

Dr. W. Killing has for the second time been awarded the Lobachevski prize of the Physico-Mathematical Society of Kasan.

Miss Martha McDonald, graduate student at the University of Chicago, is to be in charge of the department of mathematics at Oxford College for Women, Oxford, Ohio.

"Tables of the Exponential Function" computed to twenty places, compiled by C. E. Van Orstand, were published in the June number of the *Journal of the Washington Academy of Sciences*.

At the University of Chicago H. E. Slaught and G. A. Bliss have been promoted to full professorships in mathematics.

M. PIERRE BOUTROUX will be on the faculty of Princeton University this fall as a professor of mathematics. He is closely related to the Poincaré family.

At Stanford University W. A. MANNING has been promoted to an associate professorship in mathematics and H. F. BLICHFELDT to a full professorship.

Mr. H. N. WRIGHT, who received the doctor's degree at the University of California last May, has taken charge of mathematics in the University Extension work of the University of California.

Miss JOSEPHINE E. BURNS, who received the doctor's degree at the University of Illinois last June, has been appointed instructor in mathematics in the same institution in place of Dr. R. M. Winger, who accepted an assistant professorship in the University of Oregon.

On the twenty-fifth anniversary of President Slocum's assumption of office at Colorado College, appropriate celebration ceremonies were held at the June Commencement. On this occasion the trustees bestowed upon Dean FLORIAN CAJORI the honorary degree of Doctor of Laws. A few days later the University of Wisconsin conferred the degree Sc.D. upon Professor Cajori.

The July number of *The Monist* contains the following articles of interest to mathematicians: "The New Mechanics," by the late Henri Poincaré; "The Principle of Relativity as a Phase in the Development of Science," by Paul Carus; and "Robert Hooke as a Precursor of Newton," by Philip E. B. Jourdain.

The Eastern Association of Physics Teachers held its sixty-fourth meeting in Boston on March 29, 1913. An interesting item on the program was the report of a committee on new books. In this way important books are brought before teachers for discussion. Possibly this may be a good suggestion for meetings of mathematics associations.

The attendance of advanced students in mathematics at the University of Chicago summer quarter has been unusually large. All courses have been full but notably those in Theory of Functions and Integral Equations offered by Professor Bolza have been overflowing. The weekly meetings of the Mathematical Club have drawn large numbers. The programs have been devoted alternately to the discussion of pedagogical questions and the reading of technical papers of a research character.

During the last Summer Session of the University of California the High School Teachers of California held various sessions. Two of these were devoted to the consideration of mathematical questions. The following papers were read during these two sessions: "Mathematical troubles of the freshman," by Professor G. A. MILLER; "A broader program for advanced mathematics in secondary schools," by Dr. H. W. STAGER; "Course in plane surveying," by Professor F. S. FOOTE; Arithmetic and the introductory high school," by Miss T. BROOKMAN; "Arithmetic and bookkeeping of everyday life," by Mr. F. M. POWELL; "The irreducible minimum in high school mathematics," by Mr. C. M. TITUS.

On June 19 a statue of LORD KELVIN, subscribed for by the citizens of Belfast, was unveiled in that city by Sir Joseph Larmor. Belfast is the birthplace of Lord Kelvin. Another statue of Lord Kelvin, erected by the contributions of his fellow citizens in Glasgow and the West of Scotland, has been placed in position by the side of the new Kelvin Avenue in Glasgow and will be unveiled on October 8 next. A window in Westminster Abbey, in memory of Lord Kelvin, was unveiled on July 15.

Weekly popular lectures on mathematics were given during the summer session of the University of Wisconsin as follows: "The history of Greek geometry," by Professor L. W. DOWLING; "Soap films and their mathematical theory, with demonstrations," by Professor ARNOLD DRESDEN; "Zeno's paradoxes of the flying arrow and of Achilles and the tortoise, an historical treatment," by Dean FLORIAN CAJORI of Colorado College; "On the teaching of arithmetic," a conference conducted by Professor W. W. HART.

The May number of *L'Enseignement Mathématique*, the official organ of the international commission on mathematical teaching, contains an interesting popular article by GINO LORIA of the University of Genoa on "Excentricités et mystères des nombres." The March number of the same journal contains an article by A. N. WHITEHEAD of University College, London, on "The Principles of Mathematics in relation to elementary teaching," which discusses some vital topics such as are suggested in the following quotation: "There is an idea, widely prevalent, that it is possible to teach mathematics of a relatively advanced type—such as differential calculus, for instance—in a way useful to physicists and engineers without any attention to its logic or its theory. This seems to me to be a profound mistake. . . . I fully admit that the proper way to start such a subject as the differential calculus is to plunge quickly into the use of the notation in a few absurdly simple cases, with a crude explanation of the idea of rates of increase. . . . The habit of logical precision with its necessary concentration of thought upon abstract ideas is not wholly possible in the initial stages of learning. It is the ideal at which the teacher should aim."

The final installment of Professor Cajori's *History of the Logarithmic and Exponential Concepts* is found in this issue, together with a table of contents and some additions and corrections. This series of articles is not to be reprinted separately and can be obtained only as an integral part of Volume XX of the MONTHLY. The interest in historical studies will be continued in the October issue with the first installment of *Number Systems of the North American Indians*, by W. C. Eells of Whitworth College, Tacoma, Washington. These studies are based upon a critical examination of about 300 languages and dialects. The linguistic diversity of the North American Indians is one of the most remarkable features of world ethnology, and Professor Eells discusses their number words and numer systems in a highly instructive and interesting manner.

The annual report by *Science* concerning "Doctorates conferred by American Universities" was printed in the issue of August 22. This report shows in tabular

form the average number of doctorates of philosophy and of science conferred by each of 44 universities during the ten year period, 1898–1907, the actual numbers for each year from 1908 to 1913, and the total numbers for the sixteen year period, 1898–1913. Previous to 1908 Chicago had been in the lead; since that date Columbia has forged rapidly ahead. During the sixteen years twelve universities have conferred more than one hundred doctorates each. These are in order: Columbia, 702; Chicago, 648; Harvard, 588; Yale, 522; Johns Hopkins, 475; Pennsylvania, 406; Cornell, 374; Wisconsin, 206; Clark, 159; New York, 149; Michigan, 125; and Boston, 104. Twelve others have conferred from thirty to eighty each; and eleven others, from ten to eighteen each. The grand total is 5237 in all departments.

A similar tabulation is given for doctorates conferred in the science departments alone. Here, only eight universities have conferred more than 100 doctorates each, and Chicago is decidedly in the lead. They are: Chicago, 333; Johns Hopkins, 283; Columbia, 281; Cornell, 255; Harvard, 235; Yale, 234; Pennsylvania, 161; and Clark, 145. The grand total is 2541 in all science departments.

A third table shows the numbers for various departments during the same periods. The totals for these are: Chemistry, 682; Physics, 311; Zoology, 286; Psychology, 275; Mathematics, 248; Botany, 240; English, 192; History, 152; Geology, 151; Economics, 125; Philosophy, 121. Fifteen other departments show a total of 25 to 97 each; six others a total of 12 to 20 each, the remaining departments showing less than ten each.

The number of doctorates in mathematics averaged 12.1 for the years 1898–1907, and 21.1 for the years 1908–1913, with a grand total of 248 for the 16 years. For 1913 there were 20 doctorates in mathematics distributed as follows: Harvard, 4; Columbia and Johns Hopkins, 3 each; Boston, Michigan, and Yale, 2 each; Chicago, Illinois, Pennsylvania, and Princeton, 1 each.

IMPORTANT ANNOUNCEMENT

TO

ALL TEACHERS OF MATHEMATICS

AND

OTHERS INTERESTED IN MATHEMATICAL PROGRESS

THE AMERICAN MATHEMATICAL MONTHLY, since its reorganization in January, 1913, has endeavored to fulfill its mission as "A JOURNAL FOR TEACHERS OF MATHEMATICS IN THE COLLEGIATE AND ADVANCED SECONDARY FIELDS."

The numbers thus far issued, including September, have contained 234 pages, distributed as follows:

Historical Topics.
By FLORIAN CAJORI, JOSEPHINE BURNS, and G. A. MILLER,—71 pages.

Pedagogical Considerations.
By E. R. HEDRICK, G. A. MILLER, SAUL EPSTEEN, and H. E. SLAUGHT,—23 pages.

General Mathematical Information, including Mathematical Meetings, Notes and News, Book Reviews.
By G. A. MILLER, H. E. SLAUGHT, FLORIAN CAJORI, W. H. BUSSEY, W. W. BEMAN, R. R. SHUMWAY, L. C. KARPINSKI, H. E. COBB, H. L. RIETZ, S. LEFSCHETZ, A. C. LUNN, C. H. ASHTON, W. C. BRENKE, and W. D. CAIRNS,—44 pages.

Topics Involving a Minimum of Technical Treatment.
By W. H. BUSSEY, G. R. DEAN, E. R. HEDRICK, E. V. HUNTINGTON, J. L. COOLIDGE, R. P. BAKER, and R. D. CARMICHAEL,—32 pages.

Topics Involving Somewhat More Technical Treatment, designed to stimulate mathematical activity on the part of ambitious students and teachers.
By R. D. CARMICHAEL, W. O. BEAL, D. N. LEHMER, E. J. MILES, L. E. Dickson, T. E. MASON, E. C. MOLINA, and E. L. DODD,—38 pages.

Problems and Questions.
By H. C. FEEMSTER, W. D. CAIRNS, W. H. BUSSEY, R. P. BAKER, RICHARD P. LOCHNER, W. J. GREENSTREET, W. W. BEMAN, C. N. SCHMALL, H. E. TREFETHEN, A. L. MCCARTY, W. C. EELLS, S. LEFSCHETZ, A. H. HOLMES, L. S. SHIVELY, ELMER SCHUYLER, T. H. GRONWALL, A. M. HARDING, E. B. ESCOTT, J. F. LAWRANCE, D. F. KELLY, W. R. LEBOLD, EVA S. MAGLOTT, B. F. FINKEL, F. P. MATZ, E. T. BELL, V. M. SPUNAR, J. K. ELWOOD, T. M. BLAKSLEE, F. C. RUST, G. W. HARTWELL, J. SCHEFFER, M. E. GRABER, ELIJAH SWIFT, W. H. JOHNSON, M. A. MUZZY, J. V. COLLINS, ARTEMAS MARTIN, W. E. HEAL, CLARIBELL KENDALL, W. W. LANDIS, L. E. DICKSON, CHARLES MACAULEY, R. D. CARMICHAEL, L. C. MATHEWSON, B. F. YANNEY, and T. E. MASON,—26 pages.

Professor Cajori's "History of Logarithms," which has run serially from the beginning and is concluded in this issue, is not only a contribution of great value but it is, at the same time, an extremely fascinating piece of reading. It will not be obtainable elsewhere after the present limited number of extra copies is exhausted, and these will be supplied only to new subscribers to the MONTHLY.

Our readers are requested to make this known to their friends who might be interested.

The Subscription Price of the Monthly is only TWO DOLLARS ($2.50 in foreign countries) per volume of ten numbers containing over 300 pages.

Sample Copies may be obtained from the MANAGING EDITOR, H. E. SLAUGHT, 5548 Monroe Ave., Chicago, Ill.

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THE

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MATHEMATICAL MONTHLY

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MATHEMATICAL TROUBLES OF THE FRESHMAN[1]

By G. A. MILLER, University of Illinois.

The fact that there is generally a great gap between high-school mathematics and that of the freshman courses in our colleges and universities accounts for a considerable part of the troubles of the first year students in the higher institutions of learning. While it may not afford us any comfort to know that similar troubles exist in other countries, it is evidently desirable to acquaint ourselves with the extent of these troubles before we endeavor to explain their source or to devise means to mitigate them.

Conditions in Germany.—In the preface of his interesting book entitled *Die Erziehung der Anschauung*[2] Professor H. E. Timerding observes, in reference to conditions in Germany, that a deep gap divides the whole system of education. On the one side are the higher institutions of learning while on the other are the elementary schools. Everything is different on these two sides,—the objects, the methods, the views, the text-books; they are two completely separated worlds. The one has its center in scientific research, the other in practical education, and this separation is so complete that the results of scientific research come to the elementary teachers only through one or two intermediaries and in a very misleading form; while on the other hand, the university teachers pass by all pedagogic questions with great disdain.

Conditions in America.—Our American university student has perhaps one advantage over his German brother as a result of the fact that this one deep gap in the German system is generally replaced by two somewhat less serious ones in our system of mathematical education. One of these gaps is met on entering the freshman courses while the other is reached a few years later in the first courses of a really graduate nature. As the latter of these is encountered by a comparatively small number, whose deep interest inspires renewed courage, the former presents by far the more serious question to those interested in the

[1] Read before the Mathematics Section of the California High School Teachers Association, July 8, 1913.

[2] Published by B. G. Teubner in 1912.

welfare of our mathematical education. We shall therefore turn our entire attention to the gap at the beginning of the freshmen courses.

Some of the Causes.—It is not difficult to find some of the forces tending to widen this gap or to reopen it, if it should have been closed locally through heroic efforts. We have only to look at the company which our college and university teachers keep. They like to read books on higher mathematics and to investigate new or only partly explored mathematical regions. They like to go to meetings of mathematical investigators and to join in a language which is entirely unintelligible to the outside world. They love to go abroad to the biggest centers of mathematical lore and to expose themselves to the zeal for great new ideas whose enchanting comprehensiveness makes pigmies of the mathematical concepts which bewilder the freshman and strain his powers of comprehension.

Is it a wonder that such men as these should recoil from contact with those who are mainly concerned with the difficult problem of enlarging the horizon of young minds whose penetrating powers, though great when compared with those of the uneducated man, are trivial when compared with those which may be attained by many years of arduous application. The tastes of the college and university instructor, as indicated by the company which these men keep, cannot fail to call forth most serious misgivings on the part of those who are interested in a gradual and continuous mathematical development of the student.

The Widening Gap.—An important element in this situation is the fact that some of the influences tending to increase the difficulties with which the freshman has to contend are becoming more serious from year to year. With the increasing emphasis on graduate courses there comes a growing interest in research work, and a university instructor is often led to feel that his advancement in standing both in the university and among his colleagues is mainly based upon his activity in research. It is only reasonable to expect that such instructors should seek to affiliate themselves with the agencies which tend to promote their own advancement and to secure the approbation of their colleagues.

In view of these gaps in our system of mathematical education and the strong forces which tend to perpetuate them, we may at first be inclined to turn our attention to the means of bridging them at any cost. Conferences like this between high school and university instructors are steps along this line. The great work which has been accomplished during the last few years under the general direction of the International Commission on the Teaching of Mathematics tends to bring high school and university teachers in closer touch, and hence it helps to bridge the serious gap met by the freshman. However much good these efforts may accomplish they cannot eliminate permanently conditions arising from the nature of our educational system. There are cases where the complete cure of an evil is more objectionable than the evil itself, and the conditions under consideration may possibly belong to this category. At any rate, it seems wise to consider some of the effects of the gaps in our system of mathematical education before prescribing very drastic measures for closing them, or even for bridging them in many places.

Our Patch-work System.—In most schools we begin our mathematical education by leaving great gaps in the subject matter. For instance, algebra and geometry are developed as distinct subjects and the gap between them is gradually filled up as we proceed. If it is found feasible to start our mathematical farm by the cultivation of isolated patches, it may possibly be found feasible to have isolated patches in our methods of work and in our centers of interest. It may be that at the time the student reaches the university he ought to get into a new world with new views, new objects, new perspectives, and new mathematical ideals.

We are sometimes told that our first year of university instruction is bad because those who stood the highest in their classes in the high school do not make as good freshman records as some of those who did only mediocre work in the high school. Taken by itself this does not prove anything. The mediocre student in the high school may appear mediocre only in view of some arbitrary and unjust standards, or it may be that these standards are just at the high school age but would not be just later in life. The qualities leading to distinction in youth do not always lead to equal distinction in manhood. Adaptation is one of the most important faculties to cultivate, and if the student learns, at the beginning of his college course, the hard lesson that he has sometimes to adapt himself to new and radically different conditions, he has learned one of the most important lessons of life. For instance, young people do not grow gradually into the married state but they are required to adapt themselves at one step. Discontinuity is common in life and we may naturally expect to meet it in education.

Inherent Difference.—Before trying to determine whether the extensive freshman fatalities are due to high school or to college methods of instruction let us agree that these two methods ought to be different. After agreeing on this point we still have the main difficulty to solve; namely, how great this difference should be. On this point, we can probably not expect to reach perfect agreement. The more advanced the work is the more the individuality of the teacher does and ought to assert itself. Hence we should not expect as much uniformity among the methods used by freshman instructors as among those employed by the high school teachers. At this point we reach again the perplexing question how great this difference ought to be.

While my arguments thus far have been directed mainly towards placing the freshman situation in a charitable light, I would not desire to be classed with those who are entirely satisfied with the situation and who see no room for improvement. Without trying to turn high school teachers into university teachers or vice versa, I would urge most strongly that these two classes of teachers should regard it as one of their most important duties to learn from each other. The tendency to grow away from each other is exceedingly strong, and a conscious and repeated effort to overcome this tendency is necessary, in most cases, in order to maintain ourselves in a condition to render the best service. There is much that we can learn from each other.

Higher Ideals Needed.—The high school teachers, as a class, need higher scholarly ideals. The high school teachers' meetings should have more papers dealing with modern mathematical subjects, and the high school teachers' library should contain more books and journals dealing with advanced mathematics. A teacher who has ceased to hunger for more knowledge about his subject has no business in the high school. Let us not deceive ourselves. Our students know where our interests lie, and how can we expect to inspire them with zeal in the subject matter if we have not sufficient interest in it to work ahead to the best of our abilities and in accordance with our opportunities.

Mathematical Journals.—One of the chief incentives to come to a meeting of this kind should be the opportunity it gives us to learn, by consultation with each other or by means of the excellent library at this place, more about some mathematical questions. The mathematical journals should be one of the chief centers of interest, since they are of such varied grades as to come within the reach of all. At this time high school teachers should have especial interest in the improvement of the AMERICAN MATHEMATICAL MONTHLY, which is peculiarly their own journal in this country, from the present point of view. The *Mathematical Gazette* of England has somewhat similar aims.

Those who read foreign languages and who desire a journal of about the same grade in such a language, would probably find the following (in French, German, Italian, and Spanish, respectively) very instructive: *L'Enseignement Mathématique*, *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht*, *Periodico di Matematica*, and *Revista de la Sociedad Matemática Española*. Some of the other foreign languages which are less extensively read by Americans have secondary mathematical journals. For instance, Japan has such a journal which is known as "The XY" and is published in the Japanese language by the XY House. A brief review of this journal is found in the *Mathematical Gazette* for January, 1913, page 17.

Interest in Pedagogical Questions.—While the high school teacher can do much towards decreasing the gap at the beginning of the freshman year by taking a deeper interest in the subject matter and in the fields which lie just beyond those with which he is primarily concerned, the university instructor of freshman mathematics should not forget that the strong temptations to grow away from pedagogic interest must be partially counteracted. Some of our European colleagues are setting us conspicuous examples along this line. The fact that the office of President of the Mathematical Association of England, an association of teachers and students of elementary mathematics, has been filled in recent years by such eminent men as A. R. Forsyth, B. G. Mathews, G. H. Bryan, H. H. Turner, E. W. Hobson and Sir George Greenhill, illustrates the attitude of some of the most prominent English mathematicians with respect to the work of the secondary teachers.

A few of the leading American mathematicians are also taking a deep interest in high school mathematics, but I do not know of any American association of teachers of elementary or secondary mathematics which can point to a list of

such eminent successive presidents as the one which has just been cited. Whether this is due to a lack of appreciation of scientific eminence on the part of our teachers of secondary mathematics, or whether it is mainly due to the fact that too few eminent American mathematicians are interested in elementary and in secondary mathematics, may be difficult to determine. Probably both of these undesirable conditions have their influence and tend to strengthen each other. At any rate, our leading mathematicians and our teachers of secondary mathematics should be in closer touch with each other.

Great Learning and Simple Language.—The teachers of elementary mathematics are sometimes afraid of the university teachers because the latter use a language of their own. This is too often true, but it is not true of all the really great men. Great *learning* tends towards a simple language while great *cramming* tends toward an unintelligible jargon. If a man cannot talk so as to be understood you can be almost sure that he has read much but understood little, or that he has dreamed himself into many problems but did not take the time to solve any of them completely. Such dreamers may be helpful to the research student but they have little for the busy teacher.

On the other hand, the great mathematician who has really mastered many of the problems of advanced mathematics should be able to talk most clearly about the difficult mathematical questions which confront the teacher of secondary mathematics. Hence these teachers should seek great mathematicians as presidents of their associations in order that they may be able to learn new things easily at the meetings, and thus be enabled to present subjects more clearly to their students.

Among the subjects which are at the present time of common interest to many advanced mathematicians and to high school teachers, the foundations of geometry and the theories of aggregates and groups deserve especial mention.

Special Class Preparation Needed.—The instructor of the freshmen students of our colleges probably fails more frequently on account of a lack of a complete mastery of the subject which he is to teach than on account of the fact that his methods differ from those of the high school. The student expects and welcomes a change of method. He likes to get into a new atmosphere. He welcomes the greater freedom and the greater individuality of methods which he finds in college. The great disturbing element often is a lack of clearness and simplicity. This may be due to the fact that his instructor was thinking along an entirely different line until five minutes before the class. What we need most of all is a deeper interest in the subject matter on the part of our teachers, and when our teachers' conferences come to be dominated more completely by this thought, our university instructors and high school teachers will have more in common, and the gap at the beginning of the university course will become the freshman's delight instead of his stumbling block.

Resumé.—The substance of our argument is, in brief, as follows: The scientific interests of the high school teacher and the university instructor should have more in common. As the latter have generally better opportunities and more

time for study, the former should more frequently invite the latter to assist them in the solution of their more difficult scientific questions. On the other hand, the university instructor should realize that his freshman classes deal with subjects which he himself understands imperfectly, and he should be interested in arriving continually at clearer views in regard to these subjects and the methods of presenting them.

The Investigator and the Teacher.—In recent years there has been considerable discussion in reference to making a public distinction between the investigator and the teacher on the university faculty. Full professors of mathematics have been appointed, in excellent universities, mainly on the ground of their success as teachers. The question might be raised whether it would not be wiser to insist that every successful university instructor must be an investigator, but at the same time to admit that investigation in *elementary* subjects is also important, and especially in respect to the teaching of these subjects. Those who are investigating in the fields of freshman and high school mathematics should, and in many cases now actually do, fill the so-called teaching professorships.

Mathematics a Growing Subject.—One of the most harmful phrases is the assertion that "mathematics is a finished science." It would be difficult to find a shorter way to expose dense ignorance of the subject than to use this phrase seriously. Even elementary mathematics is not a finished science and indeed no one has a complete knowledge of any elementary subject.

The feeling that we are all dealing with a growing subject and that new developments in one branch are apt to throw light on other branches should awaken in us a hopeful interest in the work in the different branches. Such an interest tends to inspire us with a zeal that is contagious. This zeal is one of the greatest assets of any teacher of mathematics. It serves also as a bond which unites us all and increases our respect for the work in the various fields of mathematics. The university instructor who has great respect for his high school colleague is almost sure to be a good teacher, and the high school teacher who honors the work which his university colleague tries to do is apt to continue to grow and to inspire his students with his own zeal.

The Beauties of Mathematics.—We are all inspired by the beauties of this country when viewed from the tops of the various hills and mountains of this state. The views from this campus enthuse us and lead us to express our delight to those around us. The descriptions of the beauty of the "Golden Gate" have aroused most interesting impressions in the minds of millions of school children, as well as in the minds of adults who have never visited this part of the world. Is it not a shame that the intellectual beauties seen from intellectual heights should be passed by without also awakening our emotions? Elementary algebra and elementary geometry are largely blooming orchards. The fruit will appear later. Let us not talk continually about the fruit now. The tendency of the modern text-book is to disfigure a beautiful blooming tree by tying pictures of the fruit on its branches. I believe that more students can be inspired by the blossoms which they see than by the pictures of the fruit. At any rate, let us be

alive to the beauties of our subjects, no matter how elementary. In this way most freshman mathematical troubles will vanish notwithstanding the change of emphasis.

Let me carry this figure a little further. What I mean is that the high school teacher should have traveled widely through the beauties of mathematics and should not hesitate to talk about some of the beauties which are beyond his students. We do not hesitate to talk about the Grand Canyon of the Colorado or about the grandeur of Yosemite to those who never expect to visit these places. Why should we then hesitate to mention some of the jewels of the calculus, the grandeur of number realms, or the wonders of higher dimensions, even to those who are not sure that they will ever reach these stages. In most cases the answer is evident. We know too little ourselves about matters lying close to the subjects we teach, and therefore we cannot talk about these things in a sufficiently simple language. We have not seen the subjects ourselves but merely read about them. Teaching the most advanced subject which we have studied is like describing a country about which we have read but which we have not seen.

The Chief Essential of Good Teaching.—If I were asked to provide a formula for the manufacture of successful high school and university teachers, I should consider it necessary to include a great number of ingredients. Hence my emphasis on more scholarship *relating directly to the subject in hand* means simply that from my point of view the supply of this ingredient is, at the present time, more commonly inadequate than that of any other. If we could add a sufficiently large amount of this, together with the zeal which naturally goes with it, to the present equipment of our teachers, most of the present freshman troubles would vanish. Possibly the great prominence of athletic matters as chief subjects of conversation might even become a thing of the past.

Journals and Books.—While it is expensive to study at a university or to attend a large number of meetings, it is comparatively inexpensive to secure a few good journals and good books. Most of the best thoughts of the world find their way into journals and then into books,[1] and the most economical way of thinking is to relate ourselves to these fresh thoughts. Hence the need of good libraries is being recognized as never before, and the teacher's private library should be as complete as possible along the lines in which his work lies. As his subject connects with other realms of growing knowledge, such a library will enable him to relate his life with the sympathies and interest of all seekers of truth. This vital connection ennobles our lives as teachers and furnishes the most important element of that inspiration and sympathy between teacher and student which surmounts all obstacles.

Journals as Antidotes.—The rapid increase in the number and types of mathematical journals during the last fifty years reflects their great value. While Americans are generally quick to adopt the best, the American teachers of mathematics have been very slow in improving their efficiency by reading the mathe-

[1] Cajori's history of logarithms which has been appearing in the MONTHLY is now in the journal stage.

matical journals. The reading of the same things goes far toward creating points of contact and developing common interests, and it is these common interests between high school and university instructors which will tend to remove the objectionable features of the gap with which the freshman has to contend. When an evil in the physical world begins to assume serious proportions, its antidotes become prominent objects of study. One of the strongest antidotes against the forces which tend to produce harmful gaps in our mathematical education may be found in the uniting tendencies of the mathematical journals. As these journals constitute such a strong directing and inspiring force and provide a forum where all may meet at will for the discussion of problems, they should be heartily supported by all teachers of mathematics.

In closing I desire to lay great stress on another point, namely the history of the subjects we teach. The normal student is most deeply interested in some historical facts but most of us are too lazy or too indifferent to study the history of our subject sufficiently to speak about it in simple and accurate language. A few historical observations now and then put new life into our classes. It is all the more necessary that we cultivate a continuous acquaintance with the history of our subjects since the recent progress of our knowledge along this line has been so rapid. In history we find a bond which unites not only all those interested in mathematics but also all who are interested in the development of the intellect.

ON THE SOLUTIONS OF LINEAR EQUATIONS HAVING SMALL DETERMINANTS.

By F. R. MOULTON, University of Chicago.

The fundamental theorem respecting linear non-homogenous equations is that their solution exists and is unique if the determinant of the coefficients of the undetermined quantities is distinct from zero. But this simple theorem does not by any means contain all that is important from a practical point of view. If the coefficients of the equations are furnished by observations, or by measurements, they are determined only to a certain number of places. The question then arises how exact their solution is. In the discussion of this question the magnitude of the determinant plays an important rôle, and the results are particularly interesting and striking when the determinant is small.

For the sake of illustration of some of the peculiarities of the solutions of linear systems, before taking up the general discussion, consider the equations

$$(1)\qquad \begin{cases} .34622\,x + .35381\,y + .36518\,z = .24561, \\ .89318\,x + .90274\,y + .91143\,z = .62433, \\ .22431\,x + .23642\,y + .24375\,z = .17145. \end{cases}$$

It will be supposed that the coefficients and right members of these equations are furnished by observations, and that the possible errors to which they are subject do not exceed five units in the sixth place. The problem is to determine x, y, and z and to find the degree of accuracy of the determination.

A system of linear equations turns up in many problems of applied mathematics. Indeed, it was such systems in the theory of the determination of the orbits of the heavenly bodies from observations from the earth which led to the considerations which are developed here. They arise in all problems involving the determination of quantities by the method of least squares. In these computations it is of course essential that the work be done with numerical precision, and this has led to systematic modes of procedure and the development of control formulas for use in the process. The method, aside from the control formulas, for solving linear systems is to eliminate one unknown from all of the equations except one by use of the one. Then, one equation of the reduced set is used to eliminate a second unknown. The process is repeated until a single equation remains involving a single unknown.

In applying the process for solving linear equations, which has been described, to equations (1) no difficulty is encountered. It is true that a zero turns up at the successive stages in the first place on the left, but it is natural to multiply such equations by 10, or to add another figure on the right. It is found on carrying out the solution with seven-place tables that

$$x = -\ 1.027066, \quad y = +\ 2.091962, \quad z = -\ 0.380515. \tag{2}$$

On substituting the values of x, y, and z in (1) it is found that these equations are satisfied to the last place. Moreover, if either x, y, or z alone is changed by so much as one unit in the fifth place the equations will no longer be satisfied. But

$$\begin{cases} x = -\ 1.022773, & y = +\ 2.084125, \quad z = -\ 0.376941, \\ x = -\ 1.031229, & y = +\ 2.099457, \quad z = -\ 0.383879, \end{cases} \tag{3}$$

also satisfy equations (1) up to the last place. The variation in x is nearly one per cent., and in z nearly two per cent. That is, although equations (1) are supposed to be accurate to five places, their solution is determinate only to two places; or, expressed otherwise, the uncertainty in the result is a thousand times what might be expected.

The phenomenon exhibited by (1), (2), and (3) is to be explained. If equations (1) are written in the form

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3, \end{cases} \tag{4}$$

the expression for x is

$$x = \frac{\begin{vmatrix} d_1, & b_1, & c_1 \\ d_2, & b_2, & c_2 \\ d_3, & b_3, & c_3 \end{vmatrix}}{\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}} = \frac{D_x}{D}. \tag{5}$$

There are similar expressions for y and z.

Suppose the numerical value of D is small. If a_i, b_i, and c_i are defined only to five places and the first three places in D are zeros, D is in reality given only to two places. Consequently x is defined only to two places. In the numerical example D is small, and this is the explanation of the variations in the solutions. The value of D is

$$D = \begin{vmatrix} .34622, & .35281 & .36518 \\ .89318, & .90274 & .91143 \\ .22431, & .23642 & .24375 \end{vmatrix}.$$

Let the second column be subtracted from the first and third columns, and let the resulting third column be added to the first. These operations do not change the value of the determinant which then takes the form

$$D = \begin{vmatrix} +.00378 & .35381 & .01137 \\ -.00087 & .90274 & .00869 \\ -.00478 & .23642 & .00733 \end{vmatrix},$$

which is necessarily small because the elements of both the first and the third columns are small.

Suppose the coefficients a_i, b_i, c_i, and d_i are given by observations to a certain number of places, and consider the problem of finding the greatest variations possible in x, y, and z. Suppose the determinant D is not zero, though it may be small; then equations (4), regarding the a_i, b_i, c_i, and d_i as being exact, have a unique solution. The assumption that the coefficients are exact is equivalent to saying that all the digits beyond the number of places given are zero (assuming that all numbers are given in the decimal form). Let x, y, and z represent the solution of (4). In order to use effectively the hypothesis that the digits beyond those given are zero it will be necessary to use more places in the computations than are given in the coefficients. This was done in computing (2), and when these values of x, y, and z are substituted in (1) the digits in the sixth places are zeros.

Suppose the exact values of the coefficients are

$$(6) \qquad a_i + \alpha_i, \quad b_i + \beta_i, \quad c_i + \gamma_i, \quad d_i + \delta_i \quad (i = 1, 2, 3),$$

where a_i, b_i, c_i, and d_i are the numbers furnished by the observations, and where α_i, β_i, γ_i, and δ_i are the differences between these numbers and the true values of the quantities measured. The latter quantities are, of course, not known, but generally limits can be placed on their possible magnitudes. For example, it might be that they would not exceed five units in the first neglected place. The problem is to find the greatest admissible variations in x, y, and z for all admissible variations in the α_i, β_i, γ_i, and δ_i. For any set of α_i, β_i, γ_i, and δ_i suppose the corresponding values of the unknown quantities are

$$(7) \qquad x + \xi, \quad y + \eta, \quad z + \zeta.$$

It is supposed that the determinant is not zero with the coefficients (6).

With the notation that has been adopted the equations become

$$(8) \quad \begin{cases} (a_1 + \alpha_1)(x + \xi) + (b_1 + \beta_1)(y + \eta) + (c_1 + \gamma_1)(z + \zeta) = d_1 + \delta_1, \\ (a_2 + \alpha_2)(x + \xi) + (b_2 + \beta_2)(y + \eta) + (c_2 + \gamma_2)(z + \zeta) = d_2 + \delta_2, \\ (a_3 + \alpha_3)(x + \xi) + (b_3 + \beta_3)(y + \eta) + (c_3 + \gamma_3)(z + \zeta) = d_3 + \delta_3. \end{cases}$$

After subtracting equations (4) the equations defining ξ, η, and ζ are

$$(9) \quad \begin{cases} (a_1 + \alpha_1)\xi + (b_1 + \beta_1)\eta + (c_1 + \gamma_1)\zeta = \delta_1 - \alpha_1 x - \beta_1 y - \gamma_1 z, \\ (a_2 + \alpha_2)\xi + (b_2 + \beta_2)\eta + (c_2 + \gamma_2)\zeta = \delta_2 - \alpha_2 x - \beta_2 y - \gamma_2 z, \\ (a_3 + \alpha_3)\xi + (b_3 + \beta_3)\eta + (c_3 + \gamma_3)\zeta = \delta_3 - \alpha_3 x - \beta_3 y - \gamma_3 z. \end{cases}$$

The expression for ξ is

$$(10) \qquad \xi = \frac{\begin{vmatrix} \delta_1 - \alpha_1 x - \beta_1 y - \gamma_1 z, & b_1 + \beta_1, & c_1 + \gamma_1 \\ \delta_2 - \alpha_2 x - \beta_2 y - \gamma_2 z, & b_2 + \beta_2, & c_2 + \gamma_2 \\ \delta_3 - \alpha_3 x - \beta_3 y - \gamma_3 z, & b_3 + \beta_3, & c_3 + \gamma_3 \end{vmatrix}}{\begin{vmatrix} a_1 + \alpha_1, & b_1 + \beta_1, & c_1 + \gamma_1 \\ a_2 + \alpha_2, & b_2 + \beta_2, & c_2 + \gamma_2 \\ a_3 + \alpha_3, & b_3 + \beta_3, & c_3 + \gamma_3 \end{vmatrix}}.$$

The problem is to find the greatest numerical value of ξ for admissible values of the α_i, β_i, γ_i, and δ_i.

The denominator determinant must have the same sign for all admissible values of the α_i, β_i, and γ_i in order that the original problem shall be at all determinate, for otherwise ξ is not bounded either in the negative or the positive direction. Suppose the α_i, β_i, and γ_i are so small that terms of the second degree in these quantities are relatively insensible; then the denominator of (10) becomes

$$(11) \qquad D_1 = \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1, & b_1, & c_1 \\ \alpha_2, & b_2, & c_2 \\ \alpha_3, & b_3, & c_3 \end{vmatrix} + \begin{vmatrix} a_1, & \beta_1, & c_1 \\ a_2, & \beta_2, & c_2 \\ a_3, & \beta_3, & c_3 \end{vmatrix} + \begin{vmatrix} a_1, & b_1, & \gamma_1 \\ a_2, & b_2, & \gamma_2 \\ a_3, & b_3, & \gamma_3 \end{vmatrix}.$$

The first of these determinants, which is D of equation (5), is small in case of the equations which arise in the determination of orbits because the three columns are approximately equal. The numerical example with which this discussion was started illustrates the matter. When this condition is not satisfied initially it can be brought about, in case D is small, by linear combinations of the rows of the determinant. Hence it will be supposed that this condition is satisfied, and further that the elements of the second column are greater than those of the first, and that those of the third column are greater than those of the second.

The expansion of the second determinant of (11) is

$$\alpha_1(b_2c_3 - b_3c_2) - \alpha_2(b_1c_3 - b_3c_1) + \alpha_3(b_1c_2 - b_2c_1).$$

Let the numerically largest of the three parentheses be represented d_2. Then the second determinant of (11) can not exceed $3\delta d_2$ in numerical value, where δ is the greatest numerical value the α_i can take. There are corresponding results for the third and fourth determinants of (11). Let d be the value of the numerically greatest of d_2, d_3, and d_4. Then the sum of the last three determinants of (11) can not exceed in numerical value $9\delta d$. The value of δ is five units in the first neglected place. Consequently 9δ is about five units in the last place. But because of the approximate equality of the columns of the determinant d is also a small quantity. In the given numerical example, which is comparable to what often arises in practice, d is less than .02. In this example $9\delta d$ is less than one unit in the sixth place, while D, the first determinant of (11), is more than fifty times as great. In all such cases where D has one or more significant figures which are distinct from zero, the first significant figure of ξ is independent of the α_i, β_i, and γ_i so far as they appear in the denominator of (10). When this condition is not satisfied the problem is practically indeterminate and further discussion is useless. When this condition is satisfied the difference of the second and third columns of the numerator of (10) will have at least one significant figure which is distinct from zero, and consequently in these columns the β_i and γ_i can be neglected. Therefore when D has one or more significant figures distinct from zero, at least the first significant figure of ξ is determined by

$$(12) \qquad D\xi = \begin{vmatrix} \delta_1 - \alpha_1 x - \beta_1 y - \gamma_1 z, & b_1, & c_1 \\ \delta_2 - \alpha_2 x - \beta_2 y - \gamma_2 z, & b_2, & c_2 \\ \delta_3 - \alpha_3 x - \beta_3 y - \gamma_3 z, & b_3, & c_3 \end{vmatrix}.$$

The numerical value of ξ is greatest when the α_i, β_i, γ_i, and δ_i have such signs that all the terms of the expansion of (12) have the same sign and their largest possible numerical values. Suppose the α_i, β_i, γ_i, and δ_i all have the same greatest

possible numerical value δ. Let $|x|$, $|b_2c_3 - b_3c_2|$, $\cdots$ represent the numerical values of x, $b_2c_3 - c_3c_2$, $\cdots$; then the greatest numerical value of ξ is

$$(13)\quad \xi = \frac{\delta}{|D|}\left\{1 + |x| + |y| + |z|\right\}\left\{|b_2c_3 - b_3c_2| + |b_3c_1 - b_1c_3| + |b_1c_2 - b_2c_1|\right\},$$

which is positive or negative according as δ is taken positive or negative.

There are expressions corresponding to (13) which give limits on the numerical values of η and ζ. They differ from (13) only in that b_i is replaced by a_i and c_i by a_i respectively. Since the signs of the α_i, β_i, γ_i, and δ_i were all determined in making ξ a maximum, in general η and ζ will not also have their greatest numerical values. Necessary and sufficient conditions that ξ, η, and ζ shall simultaneously have their maximum numerical values are that the ratios

$$\begin{gathered}(b_2c_3 - b_3c_2) : (b_3c_1 - b_1c_3) : (b_1c_2 - b_2c_1),\\ (a_2c_3 - a_3c_2) : (a_3c_1 - a_1c_3) : (a_1c_2 - a_2c_1),\\ (a_2b_3 - a_3b_2) : (a_3b_1 - a_1b_3) : (a_1b_2 - a_2b_1),\end{gathered}$$

shall have the same signs in the three lines.

It is obviously improbable that in any physical problem the errors in the observations should all have the precise signs and numerical values to make ξ, η, or ζ a maximum numerically. There is a temptation to draw from this fact an erroneous conclusion, viz., that it is probable that ξ, η, and ζ should be near their mean values, or zero. This would be true if they were determined by a series of measurements whose numerical extremes are given by (13) and the two corresponding equations. But the hypothesis is that the observations give no presumption whatever as to the values of the α_i, β_i, and γ_i within the specified limits. This is actually the case in physical measurements. If many measurements are made the limits on the α_i, β_i, and γ_i are restricted, possibly, by taking their mean; but when they have once been determined any values within them are as improbable as, and no more improbable than, any other values. Consequently, there is no presumption in any particular case that ξ is near its mean value rather than near one of its extremes. In fact, equation (12) gives the true measure of uncertainty in the determination of ξ.

Equation (13) shows how the uncertainty in x depends upon the determinant D. Other things being equal the smaller D the greater the uncertainty in the solution. In the case of the numerical example considered above, equation (13) gives a total range in ξ of twelve units in the third decimal. The results given in (3) differ by eight units in the third decimal, showing that even this large variation in the solution is not the greatest possible. The value of y is uncertain nearly to two units in the second decimal, and z to twelve units in the third decimal.

An interesting point is illustrated by the numerical example which has been under discussion. The solution of (1), regarding the digits beyond the fifth place as being unknown, is determinate only to two decimal places; yet the equations will not *in general* be exactly satisfied if x, y, and z are varied in the fifth decimal. Since any digits in the fifth place in the solution are permissible, the fact that a solution does not exactly satisfy the equations does not prove that it is not correct so far as it is determinate; and the fact that it does exactly satisfy the equations does not prove that it is correct beyond where it ceases to be determinate. In general, when the determinant of a linear system of equations is small, the test of accuracy of the solution by direct substitution loses much of its value.

The determinant of the coefficients of a linear system of equations has a geometrical meaning which makes intuitionally clear the fact that when it is small the solution is poorly determined. For simplicity, consider the three equations (4). Each is the equation of a plane, and their common point of intersection is the solution of the system of equations. The direction cosines of the perpendiculars to these planes are proportional to the coefficients of x, y, and z. When the equations are multiplied by such factors that $a_i^2 + b_i^2 + c_i^2 = 1$, the direction cosines are a_i, b_i, and c_i. Suppose this has been done and consider the three perpendiculars which pass through the origin. They intersect the unit sphere whose center is the origin in points whose coördinates are the a_i, b_i, c_i ($i = 1, 2, 3$). The determinant D is then, by solid geometry, six times the volume of the tetrahedron whose four vertices are the origin and the three points a_i, b_i, c_i. The volume of the tetrahedron is small if the three points are close together, or if they are nearly on the same great circle on the unit sphere. In the first cases the three planes are nearly parallel, when obviously small uncertainties in their precise positions correspond to large uncertainties in the point at which they intersect; and in the second case the line of intersection of any two of them is nearly parallel to the third, so that again their point of intersection is poorly determined.

There remains finally the question whether the fact that the solution is determinate to a smaller number of places than the original coefficients are given may not make it possible to shorten the computation by arranging the formulas so that tables with a small number of places can be used without loss of accuracy. If the solution is made by successive elimination of the unknowns the tables must contain at least as many places as there are significant figures in the coefficients, for significant figures are lost in the process by the addition of nearly equal quantities having opposite signs. But if the solution is made by determinants, the elements of one or more columns can be made small by simple linear combinations of the columns. Since the expansion of a determinant is made up of terms having a factor from each column, each term contains one or more small factors. In multiplying together several factors containing various numbers of significant figures, it is sufficient to use tables having one more place than appears in the smallest factor. Hence short tables can be used if one or more columns of the determinants have only small numbers.

The determinant solution of (1) for x is

$$x = \frac{\begin{vmatrix} .24561, & .35381, & .36518 \\ .62433, & .90274, & .91143 \\ .17145, & .23642, & .24375 \end{vmatrix}}{\begin{vmatrix} .34622, & .35381, & .36518 \\ .89318, & .90274, & .91143 \\ .22431, & .23642, & .24375 \end{vmatrix}}.$$

If in the numerator determinant two thirds of the third column is subtracted from the first column, and if the second column is subtracted from the third, the value of the determinant is not altered. In the denominator let the middle column be subtracted from each of the others, and then the third column added to the first. The resulting expression for x is

$$x = \frac{\begin{vmatrix} .00216, & .35381, & .01137 \\ .01671, & .90274, & .00869 \\ .00895, & .23642, & .00733 \end{vmatrix}}{\begin{vmatrix} +.00378, & .35381, & .01137 \\ -.00087, & .90274, & .00869 \\ -.00478, & .23642, & .00733 \end{vmatrix}}.$$

The computation can now be made with four-place tables containing all the numbers required on two pages. In computing x it is found in two or three minutes that to three figures its value is $-$ 1.03, which is as far as it is determinate.

THE UNIVERSITY OF CHICAGO,
May 19, 1913.

THE ACCURACY OF INTERPOLATION IN A FIVE-PLACE TABLE OF LOGARITHMS OF SINES.

By A. M. KENYON and G. JAMES, Purdue University.

In a table which gives the logarithms of sines for each minute of angle, the logarithms of sines of angles to tenths of minutes are obtained by interpolation.

The name *Ordinary Interpolation* will be used to denote the process based on the assumption:

(1) As the angle increases from m' to $(m+1)'$ the increment of the logarithm of its sine varies as the increment of the angle; that is,

$$\log \sin (m + n/10)' = \log \sin m' + (n/10)(\log \sin(m + 1)' - \log \sin m') \qquad (n = 1, 2, 3, \cdots 9).$$

The error introduced by ordinary interpolation arises from two sources:

(*a*) The *essential* error, due to the assumption (1).

(*b*) The *tabular* error, due to the fact that the tabulated values of log sin m' and of log $\sin(m+1)'$ are in error.

The essential error increases as the angle decreases and makes ordinary interpolation unreliable for small angles. The tabular error is irregular but nowhere in the table does it exceed, in absolute value, one unit in the fifth place.

To make this clear we need to note that whenever an irrational number is approximated by a rational number correct to n decimal places, an error is introduced which in absolute value is less than $5/10^{n+1}$. Thus if we write $\sin 45° = .70711$, the result is too large by an error between .000003 and .000004.

Let $\log \sin m' = a$, $\log \sin(m + 1)' = b$ and let $a - e_1$, $b - e_2$, be the five place values of these same logarithms. Then by ordinary interpolation (assumed to be a correct process since we seek here only the tabular error), the interpolated value of $\log \sin(m + n/10)'$ would be

$$c = a + (n/10)(b - a) = c_1 - e_3$$

where c_1 represents the five place value of c.

But by ordinary interpolation from a five place table, the result will be

$$\begin{aligned} c_2 &= a - e_1 + (n/10)(b - e_2 - a + e_1) \\ &= a + (n/10)(b - a) - [e_1 + (n/10)(e_2 - e_1)] \\ &= c_1 - [e_3 + e_1 + (n/10)(e_2 - e_1)]. \end{aligned}$$

Whence the five place tabular error is

$$c_1 - c_2 = (\bar{n}/10)e_1 + (n/10)e_2 + e_3, \text{ where } n + \bar{n} = 10,$$

and obviously the absolute value of this error is less than .00001.

Actual trial of a very large number of cases (over 1,200) shows that ordinary interpolation below 2° gives the result correct to five places in less than 30 per cent. of all cases; that it gives an error of one in the fifth place in about 50 per cent., and an error of two or more in the fifth place in about 20 per cent. of cases. There is no case above 1° 30′ where the total error is greater than one unit in the fifth place. Above 2° it gives the result correct to five places in from 70 per cent. to 83 per cent. of all cases and in no case tried is the error greater than one unit in the fifth place.

The name *Logarithmic Interpolation* will be used to denote the process based on the following assumption:

(2) For sufficiently small angles, the sine varies as the angle; which gives, if m and n denote measures of angles in the same unit,

$$\log \sin(m + n/10) = \log \sin m + \log(m + n/10) - \log m.$$

Here the essential error decreases with the angle; the tabular error as before is irregular but nowhere in the table exceeds, in absolute value, one unit in the fifth place.

Actual trial of all cases (to tenths of minutes) shows that in no case below 5° 45′ does logarithmic interpolation introduce a total error greater than one in the

fifth place.[1] Below 5° it gives the result correct to five places in about 60 per cent. of all cases and shows an error of one in the fifth place in about 40 per cent. of cases.

Below 2°, logarithmic interpolation is certainly preferable to ordinary interpolation for three reasons: (1) it gives the correct result in a larger number of cases; (2) it gives an error in the fifth place in a smaller number of cases; (3) it never gives an error greater than one in the fifth place.

Above 2°, ordinary interpolation is preferable for the same three reasons. All cases between 1° 30′ and 2° 15′ were examined.

An examination of 200 cases in the neighborhood of 84° shows that ordinary interpolation will give the result correct to five places in about 83 per cent., and will show an error of one in the fifth place in 17 per cent. of cases.

Below is a table of results of the statistical investigation.

Interval of Angle.	Number of Cases Examined.	Per Cent. of Cases Giving Result Correct to 5 Places.		Per Cent. of Cases Showing an Error of One in the 5th Place.		Per Cent. of Cases Showing an Error of Two or More in the Fifth Place.	
		Ordinary.	Logarithmic.	Ordinary.	Logarithmic.	Ordinary.	Logarithmic.[2]
1′–1°	45	Fails	60	Large	40		0
1° –1° 30′	97	33		57		10	0
1° 30′–1° 45′	All	55	63	45	37	0	0
1° 45′–2°	All	63	69	37	31	0	0
2° –2° 15′	All	71	62	29	38	0	0
2° 15′–3°	55	78	60	22	40	0	0
3° –4°	55	80		20		0	0
4° –5°						0	0
5° –5° 45′	All		66		34	0	0
5° 45′–6°	All		64		36		[3]

A THEOREM ABOUT ISOGONAL CONJUGATES.

By DAVID F. BARROW, Harvard University.

As an introduction let us recall two well-known theorems of elementary plane geometry:

THEOREM I. *Given a triangle $A_1A_2A_3$ (Fig. 1) and any point P not a vertex, join P to the three vertices and reflect each of the three lines thus drawn in the bisector of the angle at the corresponding vertex. The three reflected lines will meet in a point P', called the isogonal conjugate of P.*

THEOREM II. *Given a triangle and a point marked at random on each side. If three circles be drawn, one through each vertex and the two adjacent marked points, these three circles meet in a common point.*

[1] The first error of two in the fifth place enters at 5° 53′.7

[2] All cases examined for this column.

[3] At 5° 53′.7 the first error of 2 in the fifth place enters.

We propose to prove a new theorem:

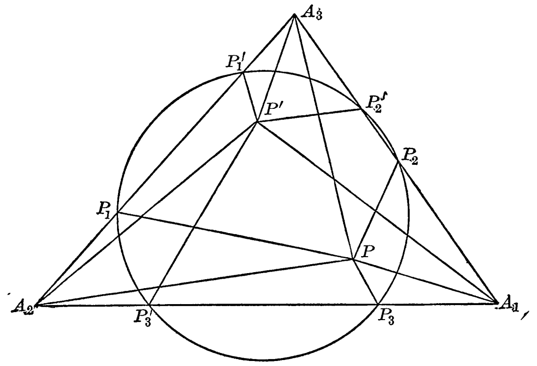

FIG. 1.

THEOREM III. *Given a triangle* $A_1A_2A_3$ *(Fig. 2) and a circle cutting each side in two points,* P_1, P_1'; P_2, P_2'; P_3, P_3', *where* P_i *and* P_i' *are on the side opposite* A_i. *If a point* P *be located as the common intersection of the three circles through the three sets of points* $P_iA_jP_k$; *and a point* P' *as the common intersection of three circles through* $P_i'A_jP_k'$; *then* P *and* P' *are isogonal conjugates.*

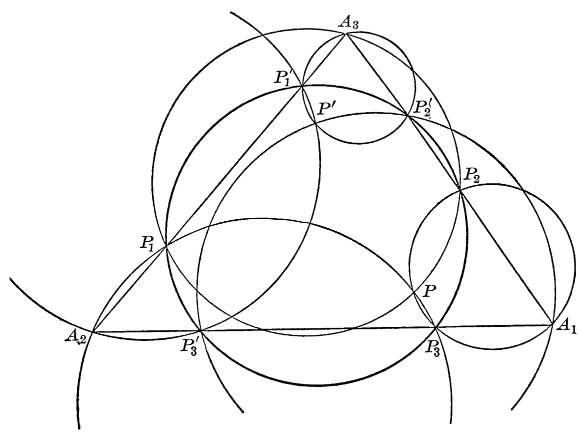

FIG. 2.

Lemma. Let P and P' be isogonal conjugates with regard to the triangle $A_1A_2A_3$ (Fig. 1).

Choose three points P_1, P_2, P_3, one on each side of the triangle, so that

$$\measuredangle PP_1A_2 = \measuredangle PP_2A_3 = \measuredangle PP_3A_1 = \theta$$

and choose three other points P_1', P_2', P_3', one on each side, and such that

$$\measuredangle P'P_1'A_3 = \measuredangle P'P_2'A_1 = \measuredangle P'P_3'A_2 = \theta$$

where θ is any angle. Then we can show that these six chosen points all lie on a circle.

From the similar triangles PA_1P_3 and $P'A_1P_2'$

$$\frac{A_1P_3}{A_1P} = \frac{A_1P_2'}{A_1P'}$$

and from similar triangles PA_1P_2 and $P'A_1P_3'$

$$\frac{A_1P_3'}{A_1P'} = \frac{A_1P_2}{A_1P}$$

Multiply these two equations together, canceling denominators.

$$A_1P_3 \times A_1P_3' = A_1P_2 \times A_1P_2'.$$

Hence P_2, P_2', P_3 and P_3' are four concyclic points. In like manner P_1, P_1', P_2, P_2', and also P_3, P_3', P_1, P_1' are concyclic. This gives us, apparently, three circles on each of which lie four of the six points. Now the sides of the triangle are the three radical axes of these circles taken two at a time. All three circles cannot be distinct because these three radical axes do not concur in a point. Hence at least two circles coincide, and so all six points are concyclic.

Observe also that by the construction of Fig. 1

$$\measuredangle PP_iA_j + \measuredangle PP_kA_j = 180°$$

and so P, P_i, A_j, P_k are concyclic. Therefore P is the point where the three circles $P_iA_jP_k$ meet. Similarly P' is the intersection of the three circles $P_i'A_jP_k'$.

We get the proof of Theorem III as follows:

Suppose we start with the triangle $A_1A_2A_3$; put down a circle cutting its sides in points P_1, P_1'; P_2, P_2'; P_3, P_3'; and locate P as the point common to the three circles $P_iA_jP_k$. Then the lemma shows that the isogonal conjugate of P lies on each of the three circles $P_i'A_jP_k'$; and since only one point does this, then the isogonal conjugate of P is that point.

If we interchange two of the points that lie an a side, say P_1 and P_1', we get a new pair of isogonal conjugates. Evidently four different pairs of isogonal conjugates can be obtained in this way, all determined by the same given circle cutting the sides. If each pair of isogonal conjugates be connected by a straight line and the perpendicular bisectors of these four lines be erected; then these four bisectors all meet in the center of the given circle. The only proof of this, which I have been able to devise, is rather tedious; but doubtless a simple and elegant proof exists.

If the given circle is tangent to one side of the triangle, but cuts the other two in distinct points, then it determines only two pairs of isogonal conjugates. If it be tangent to two sides it determines but one pair of isogonal conjugates. If it be an inscribed or escribed circle it determines one self-conjugate point, its center.

SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

By H. E. SLAUGHT, University of Chicago.

The twentieth summer meeting and the seventh colloquium of the American Mathematical Society were held at Madison, Wisconsin, during the week September 8–13, 1913. There were 57 members of the Society in attendance at the meetings, together with a considerable number of others who were directly or indirectly interested. The summer meeting of the Society is the most representative in attendance of any during the year. Professor H. F. Blichfeldt from Stanford University and Professor W. F. Osgood from Harvard University represented the far West and the far East; while Professor G. H. Ling from the University of Saskatchewan, Canada, and Professor Oscar Bolza from the University of Freiburg, Germany, added an international character to the gathering. The largest delegation was naturally from the University of Wisconsin on whose campus the meetings were held. Ten members of the Wisconsin staff were in attendance. The next largest delegation was from the University of Chicago with seven members of the faculty and three graduate students. Then followed the University of Minnesota with five members, the University of Illinois with four, Indiana University with three, Harvard University, Brown University, Northwestern University and the University of Michigan with two each, Stanford University, University of Iowa, University of Freiburg, University of Missouri, University of Saskatchewan, University of Cincinnati, Washington University, Dartmouth College, James Millikan University, Pennsylvania State College, Miami University, University of Oregon, and University of Wooster with one each. Important institutions in the East from which there was no representation were Yale, Columbia, Princeton, Cornell and Pennsylvania; and in the West, Kansas, Nebraska and California.

Conspicuous for their absence also were representatives of the many colleges, especially in the Middle West, where the distances were not insuperable. In this connection it may be said with some force that the usual hasty reading of brief outlines of highly abstract and technical papers has little interest for the average teacher of mathematics in the colleges, and hence the inducement is small for him to attend such meetings. It may well be that the Society has gone too far in confining its meetings, in general, to such a presentation of papers and that, even for the members who do attend, the more frequent presentation of expository papers would be welcomed. At any rate, those who attended the Colloquium were charmed with the clearness of exposition on the part of both speakers. Even those who had little knowledge of the fields covered were most agreeably surprised to find that they were able to understand practically all of the matter presented. This revelation and the further phenomenon of a similar nature with respect to a paper on the general program, by Professor Bôcher of Harvard University, on "The infinite regions of various geometries," which had been

scheduled for ten minutes but for which Professor Osgood, in presenting it, used forty minutes, to the great delight and satisfaction of all who heard him,—these experiences led many to declare on the spot that time should be given in all our programs for the presentation in detail of some expository papers.

The general program, including four half day sessions, contained 46 papers, several of which were read by title only, the authors in such cases not being present. The time allowance for the papers read averaged fifteen minutes each. It is impossible to present any details of a technical paper in this time. Those who attempt to do so usually fail even to make clear the salient points of their papers; but those who devote the time allotted to stating the historical setting, the nature of the problem which the paper is attacking, and the results reached, usually leave a clear impression on the minds of the auditors. On this occasion there was a larger number than usual who followed the latter method of presentation, but some still persisted in attempting the impossible. Even ten minutes used in telling what a paper is about and what conclusions are reached is better than twenty or twenty-five used in attempting to show details of proof for a paper which may have required a year in preparation.

But, after all, the most important feature of such meetings is the opportunity to meet the men who are doing things in mathematics. To spend a week in intimate social intercourse with the choice men of our profession is the best tonic for scientific activity which one can take. The Madison meetings were admirably arranged to provide a maximum of opportunity for this purpose. There was ample time in the mornings, at noontime, and in the afternoons and evenings for little gatherings at the University Club House, in the homes of the local members, in the libraries, and on excursions, to renew old acquaintances and to make new ones, to strengthen personal attachments and to gain inspiration from close touch with others. There were dinner groups at the homes of Professors Van Vleck, Mason and others; there was an automobile ride about the beautiful suburbs of Madison in machines owned by the local members and their friends; there was a tour of inspection about the campus and buildings of the University of Wisconsin; and there was an excursion on Lake Mendota in a specially chartered steamer which provided a two hours' social gathering of all the members, while the beautiful views of the Capitol, the University, and the highland borders of the lake formed a panoramic background. The boat ride ended at the Golf Club House where dinner was served and informal addresses were made, some in lighter vein, some in serious consideration of matters important to the Society. The one thing lacking was the presence of Professor F. N. Cole whose services as Secretary of the Society and Managing Editor of the *Bulletin*, for many years, have led us to think that a meeting can hardly be complete without him. In recognition of this universal sentiment a telegram was sent to him expressing regret for his enforced absence.

As intimated above, the colloquium was the great attraction of this meeting. There were five lectures by Professor W. F. Osgood, of Harvard University, on "Topics in the theory of functions of several complex variables," and five lectures

by Professor L. E. Dickson, of the University of Chicago, on "Certain aspects of a general theory of invariants, with special consideration of modular invariants and modular geometry." It was a great satisfaction to listen to two such masters of exposition as Professors Osgood and Dickson and to have the results of long investigation in important fields so clearly set forth that even non-specialists in those fields could follow and understand at least sufficiently for real enjoyment. These lectures will doubtless be published and will thus become available as important contributions in their respective fields.

Six other colloquia have been held by the Society at intervals of three or four years and thus a body of mathematical investigations is gradually forming, as one product of our own membership, which is unique in character and importance. The attendance of 51 auditors at this colloquium exceeded that of any previous one, the nearest approach being that of the Yale colloquium in 1907 when there was an attendance of 43. There were 25 at Ithaca in 1901, 31 at Boston in 1903, and 28 at Princeton in 1909. The Madison colloquium was also marked as the only one thus far held in the West. Indeed, it was the "farthest west" summer meeting except the one in St. Louis in connection with the World's Exposition in 1904.

BOOK REVIEWS.

W. H. Bussey, Chairman of the Committee.

College Algebra. By William Benjamin Fite. D. C. Heath & Co., Boston, 1913. iv + 283 pages. $1.40.

The book covers the usual topics of elementary algebra in about the conventional order. In general, the treatment is lucid and the proofs are as rigorous as the limited comprehension of young students will permit. There are nineteen chapters on the following topics: The fundamental operations, factors and multiples, fractions, linear equations in one unknown, systems of linear equations in two or more unknowns, fractional and negative exponents and radicals, quadratics, systems of equations in two unknowns solvable by means of quadratics, progressions, permutations and combinations, mathematical induction, complex numbers, theory of equations, determinants, inequalities, partial fractions, logarithms, variation, and infinite series.

In Chapter I the fundamental operations are defined geometrically; the assumptions concerning real numbers are definitely stated, and the consequences drawn as theorems. While the teacher may find this treatment desirable, it will probably prove of little real value to those students for whom the book is intended.

In Chapter V one method of solution for systems of linear equations in two or more unknowns is by the use of second and third order determinants as formulae. Those teachers for whose purposes such an elementary treatment of

determinants is sufficient may omit Chapter XIV in which determinants are more extensively treated. In Chapter XI the treatment of mathematical induction is particularly commendable for the way in which it emphasizes both parts of the proof; namely, the direct verification of the theorem for the smallest admissible value of the integer, and the proof that if the theorem is true for one value of the integer, it is true for the next greater value.

In Chapter XII the concrete geometrical treatment of complex numbers is noteworthy. However, for students who have not studied trigonometry, articles 134–138, which involve the polar representation of complex numbers and their consequences, must be omitted.

In the last chapter the treatment of infinite series does not seem to belong to such an elementary text. It would seem preferable to omit it, and give a little broader discussion of "Limit," in Chapter IX, in treating infinite geometric series. The author's omission of the symbol for infinity and the word infinity itself, except as it is implied in the phrase "infinite series," in order to avoid the misinterpretation of ∞ as a definite, fixed number, seems very undesirable. The symbol is so convenient and is used so much in all mathematical literature, that the remedy against its misinterpretation is not to omit the symbol or word, but to retain it and give it the proper meaning.

The introduction of some of the principles and problems of analytic geometry is a desirable innovation in this text where graphical methods are used freely. In general, the book is by no means radical either in treatment or subject matter, and it will probably find favor with many teachers.

H. L. Slobin.

PROBLEMS AND QUESTIONS.

Remarks by the Managing Editor.

The solutions of problems have not come in as promptly during the vacation season as at other times of the year. Hence none are given in this issue, those on hand being held over till November, when it is hoped that a full supply will be ready.

An examination of the problems proposed during this year, and of the solutions given, reveals the fact that, for the most part, solutions are not forthcoming until several months after the problems are proposed, showing that there is an *abiding interest* among the readers of the Monthly in these problems and that, even when solutions are not sent in immediately, they may still be expected later on.

Below is given an analysis of the situation so far as concerns the present year. We are glad to publish solutions of problems whenever they may have been proposed, but we are especially anxious to clear up by December, if possible, those thus far proposed this year, and we would urge all who are interested in this department to send in solutions to the Committee as promptly as may be.

(4) Please be as *concise and brief* as is consistent with clearness, omitting the more obvious details but emphasizing the important steps.

(5) Please write *clearly and on one side only* of a sheet of paper and put *not more than one* solution on the same sheet.

In conclusion, attention is called to the statement made in the January issue under the heading "Problems and Questions," where the hope was expressed that in time this department might become useful in throwing light upon real difficulties arising in connection with actual work in which we are engaged. Some problems proposed are of this character and it is hoped that still more may be in this line. In order to encourage such questions, in cases where it is not convenient to formulate a problem, and where the questioner may not care to advertise his difficulty, the editors will provide a section of miscellaneous questions and answers (withholding the names of both questioners and answerers when so requested). Such questions might well refer, for example, to the selection of subject matter and method of presentation in courses in collegiate mathematics, and they might elicit replies in the form of a short symposium or they might even lead in some cases to extended articles. On the other hand, these questions might refer to difficulties of interpretation or to obscure steps in some demonstration or to anything whatsoever of mathematical import on which the questioner honestly seeks light. We may not be able to answer all such questions, but, at least, we will try to do so or to find some one who can. Communications for the department of "Miscellaneous Questions" may be sent directly to the Managing Editor, or preferably to Professor R. D. Carmichael, Indiana University, Bloomington, Ind., who will have this special department in charge. Of course all such communications should be signed but the names will be withheld from publication if so requested.

NOTES AND NEWS.

Florian Cajori, Chairman of the Committee.

The attention of all is called to the remarks of the Managing Editor under the heading "Problems and Questions" of this issue.

Mr. F. E. Carr, who is instructor in mathematics in Oberlin College, will spend next year on leave of absence in graduate study at the University of Chicago.

Professor Gordon N. Armstrong, of Ohio Wesleyan University, who last March passed successfully his examination for the degree of doctor of philosophy at the Technische Hochschule in Munich, spent the second semester traveling in Europe and in visiting one or two of the more important universities in Germany.

In the *Journal of the Washington Academy of Sciences* (Vol. III, June 19, 1913) C. E. Van Orstrand publishes tables of the exponential functions, e^x, e^{-x}, for $x = 0.0$ to $x = 32.0$, to either 20 decimals or 20 (sometimes more) significant figures.

W. C. Eells, professor in Whitworth College, Tacoma, contributes to the *Bibliotheca mathematica* for July, 1913, a short article on "Numerals in Indian Languages of North America." A fuller account of this research will appear in the November issue of the Monthly.

Professor R. R. Shumway of the department of mathematics of the University of Minnesota has been appointed a member of the administrative board of the College of Science, Literature, and the Arts.

Assistant Professor Guy H. Albright, who was at Harvard University during the second semester of last year, resumes his work at Colorado College this fall.

It will be of interest to many readers of the Monthly to know that abstracts of fifteen of the thirty-six German reports to the International Commission on the Teaching of Mathematics may be found in the *Teachers College Record* of March, 1912 (published by the Teachers College, Columbia University, 30 cents per copy).

Dr. E. L. Dodd, of the University of Texas, contributes to the *Bulletin of the Mount Weather Observatory* (issued in April, 1913) an interesting discussion of the question: "Is the average of measurements the best approximation for the true value or normal value?"

The University of California Publications in Mathematics for June, 1912, contain an article by Henry W. Stager, "On Numbers which Contain No Factors of the Form $p(kp + 1)$."

An article, "The method of monodromie with applications to three parameter quartic equations" by Professor R. P. Baker, of the State University of Iowa, appeared in the March number of the *Annals of Mathematics.*

The psychology of the errors committed in elementary mathematics is the subject of an article which has appeared in recent numbers of *School Science and Mathematics.* It is written by Principal Thomas J. McCormack of the La Salle-Peru High School of La Salle, Ill.

Five candidates for the Doctorate in mathematics were awarded the degree at the autumn convocation at the University of Chicago; namely, W. L. Miser, instructor-elect at the University of Minnesota; F. M. Morrison, instructor on leave of absence from the University of Washington; W. Warren, instructor at the University of Manitoba; E. J. Moulton, just promoted to an assistant professorship at Northwestern University; and W. C. Krathwohl, associate professor-elect at Ripon College. Miss Mildred L. Sanderson, who received the degree in June, is instructor-elect at the University of Wisconsin.

Ginn and Company have recently published "The Teaching of Arithmetic" by Professor David Eugene Smith of Columbia University. The work has been prepared with a view to the needs of teachers' reading circles and of those who are giving instruction or supervising the work in arithmetic in the elementary

schools. Another book of the same class which has appeared recently is DR. A. W. STAMPER'S "Text-book on the Teaching of Arithmetic," published by the American Book Company. Dr. Stamper is head of the department of mathematics at the State Normal School in Chico, California.

The *Bibliotheca mathematica* for July, 1913, contains two short articles by L. C. KARPINSKI, the first on "Recorde's Whetstone of Witte," the second on "The Oxford Mathematician, John Caswell (1655–1712)." To the biographical detail about Caswell we add a statement from Bishop Berkeley which apparently refers to a publication of Caswell not noted by Karpinski. Speaking of the cone and the archimedean cylinder circumscribed about a sphere, Berkeley[1] says: *Idem quod Tacquetus, etiam Cl. Wallisius in additionibus et emendationibus ad cap. LXXXI. algebrae suae, a D. Caswello ope Arithmetices Infinitorum demonstratum exhibet.*

HEINRICH WEBER, professor of mathematics in the University of Strassburg and author of works on algebra, died on May 17 from apoplexy. He was president of the third International Congress of Mathematicians, which was held at Heidelberg in 1904, and he wrote a large number of mathematical articles. He is probably best known to Americans by his algebra, consisting of three large volumes, and by the Encyklopädie der Elementar-Mathematik.

The EULER-KOMMISSION has issued a circular in which attention is called to the fact that the publication of Euler's Complete Works is more expensive than had been anticipated. It appears now that the deficit might reach two hundred thousand francs, and it is proposed to organize a society whose members are to make annual contributions of at least ten francs during the period of about fifteen years required for the publication. These members are to receive annually a brief report of the progress of the publication and also portraits of Euler.

In the *Tôhoku Mathematical Journal* for May, 1913, DR. G. B. HALSTED traces the history of the prismoidal formulas. Newton (*Methodus Differentialis*, 1711; further carried out by Cotes, in the works of Newton collected posthumously, 1722) showed how an area or volume could be evaluated from parallel cross-sections, and especially that from three cross-sections, following at the same distance apart, we get approximately the enclosed segment if we add the outer sections to four times the mid-section and multiply the sum by a sixth of the distance between the outer sections. Maclaurin (1742, *Fluxions*, No. 848) showed that Newton's special rule gives the content exactly when every section parallel to the base is a function of its distance from it of a degree not higher than the third, $f(x) = n_0 + n_1x + n_2x^2 + n_3x^3$. In Halsted's *Mensuration* (1881) the applicability of Newton's three-term prismoidal formula was treated without the calculus; the term prismatoid was introduced for a polyhedron with no summits other than the vertices of two parallel faces. In the present article Halsted derives a criterion for two-term prismoidal formulas.

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[1] *The Works of George Berkeley*, Vol. III, Oxford, 1871, p. 54.

In beginning the re-publication in full of the Final Report of the National Committee of Fifteen on Geometry Syllabus, *The Irish Journal of Education* makes the following significant statement:

"To Professor Slaught, of the University of Chicago, Managing Editor of the *American Mathematical Monthly,* and Chairman of the National (U. S. A.) Committee on Geometry Syllabus, we are indebted for permission to re-publish in our JOURNAL the Final Report of the Committee. Very many copies of this Report have been called for and distributed in America—about 22,000 altogether, we think. In Ireland, we are not mistaken, we think, in saying that ours is the only body which has troubled itself with this important Report. A couple of Boards directly connected with education were unrepresented at the Congress—Is this a justification of the official assertion that there is money for everything but education? It is left to a body of hard-working, badly treated, and indifferently paid teachers, to do something to maintain educational progress, and make known the advances in other countries. Truly there is something strange in this spectacle."

IMPORTANT ANNOUNCEMENT

TO

ALL TEACHERS OF MATHEMATICS

AND

OTHERS INTERESTED IN MATHEMATICAL PROGRESS

THE AMERICAN MATHEMATICAL MONTHLY, since its reorganization in January, 1913, has endeavored to fulfill its mission as "A JOURNAL FOR TEACHERS OF MATHEMATICS IN THE COLLEGIATE AND ADVANCED SECONDARY FIELDS."

A selection from the Tables of Contents of the first eight numbers includes articles on—

The History of Mathematics, such as the following:

"History of the Exponential and Logarithmic Concepts," by PROFESSOR FLORIAN CAJORI, of Colorado College;

"The Foundation Period in the History of Group Theory," by JOSEPHINE BURNS, Graduate Student at the University of Illinois;

"Errors in the Literature on Groups of Finite Order," by PROFESSOR G. A. MILLER, University of Illinois.

Pedagogical Considerations, such as the following:

The "Foreword" concerning Collegiate Mathematics, by PROFESSOR E. R. HEDRICK, University of Missouri;

"Some Things we wish to know," by PROFESSOR E. R. HEDRICK;

"Mathematical Literature for High Schools," by PROFESSOR G. A. MILLER;

"Mathematical Troubles of the Freshman," by PROFESSOR G. A. MILLER;

"Minimum Courses in Engineering Mathematics," by PROFESSOR SAUL EPSTEEN, University of Colorado;

"Incentives to Mathematical Activity," by PROFESSOR H. E. SLAUGHT, University of Chicago.

General Mathematical Information, such as the following:

"The Third Cleveland Meeting of the American Association for the Advancement of Science," by PROFESSOR G. A. MILLER;

"Western Meetings of Mathematicians," by PROFESSOR H. E. SLAUGHT;

"Summer Meeting of the American Mathematical Society," by PROFESSOR H. E. SLAUGHT;

"Notes and News" of events pertaining to mathematics, under the direction of a committee of which PROFESSOR FLORIAN CAJORI is chairman;

"Book Reviews" and announcements of new books in Mathematics, under the direction of a committee of which PROFESSOR W. H. BUSSEY, University of Minnesota, is chairman.

Topics Involving a Minimum of Technic 〉 reatment, such as the following:

"Maximum Parcels under the New Parcel Post Law," by PROFESSOR W. H. BUSSEY;

"Precise Measurements with a Steel Tape," by PROFESSOR G. R. DEAN, Missouri School of Mines;

"A Direct Definition of Logarithmic Derivative," by PROFESSOR E. R. HEDRICK;

"A Simple Formula for the Angle Between Two Planes," by PROFESSOR E. V. HUNTINGTON, Harvard University;

"On the Solutions of Linear Equations having Small Determinants," by PROFESSOR F. R. MOULTON, University of Chicago;

"The Accuracy of Interpolation in a Five-Place Table of Logarithms of Sines," by PROFESSORS A. M. KEYNON and G. JAMES, Purdue University;

"A Theorem about Isogonal Conjugates," by DAVID F. BARROW, Harvard University;

"The Significance of the Weierstrass Theorem," by PROFESSOR E. R. HEDRICK;

"On the Impossibility of Certain Diophantine Equations and Systems of Equations," by PROFESSOR R. D. CARMICHAEL, Indiana University;

"A Computation Formula in Probability," by E. C. MOLINA, New York City;

"Two Geometrical Applications of the Method of Least Squares," by PROFESSOR J. L. COOLIDGE, Harvard University;

"A Puzzle Generalized," by PROFESSOR R. P. BAKER, University of Iowa;

"Problems Proposed and Solved," under the direction of a committee of which PROFESSOR B. F. FINKEL, Drury College, is chairman.

Topics Involving Somewhat More Technical Treatment, designed to stimulate mathematical activity on the part of ambitious students and teachers. Such articles have occupied only about one-sixth of the entire space; for example, such as the following:

"The Remainder Term in a Certain Development of $F(a+x)$," by PROFESSOR R. D. CARMICHAEL, Indiana University;

"A Geometric Interpretation of the Function F in Hyperbolic Orbits," by PROFESSOR W. O. BEAL, Illinois College;

"Certain Theorems in the Theory of Quadratic Residues," by PROFESSOR D. N. LEHMER, University of California;

"Some Inverse Problems in the Calculus of Variations," by DR. E. J. MILES, Yale University;

"Amicable Number Triples," by PROFESSOR L. E. DICKSON, University of Chicago;

"The Probability Integral," by PROFESSOR E. L. DODD, University of Texas.

Professor Cajori's "History of Logarithms" ran serially from the beginning and had covered about 60 pages when it was concluded in the September issue. This is not only a contribution of great value but it is, at the same time, an extremely fascinating piece of reading. It will not be obtainable elsewhere after the present limited number of extra copies is exhausted, and these will be supplied only to new subscribers to the MONTHLY.

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NUMBER SYSTEMS OF THE NORTH AMERICAN INDIANS.*

By W. C. EELLS, Whitworth College.

The linguistic diversity of the Indians inhabiting the North American continent is one of the most remarkable features of world ethnology.[1] The late director of the Bureau of American Ethnology[2] says: "In philology North America presents the richest field in the world, for here is found the greatest number of languages distributed among the greatest number of stocks." The Bureau recognizes almost three score distinct linguistic families having no lexical resemblance, no apparent unity of origin, no relation to European or Asiatic languages. These "families" are further subdivided linguistically into 750 "tribes" or languages.[3]

* Presented to the American Mathematical Society, San Francisco Section, Oct. 20, 1913.

[1] *Bibliographical Note.*—It is impossible to give a comprehensive bibliography of this subject in a small space. Most of the material is in hundreds of separate vocabularies, grammars, dictionaries and discussions of the various languages of the American Indians. Considerable may be found in the reports of the Bureau of American Ethnology, Washington, D. C. Pillings Bibliographies, published by this Bureau from 1887 to 1894, contain references to most of the literature up to the date of their publication for the nine most important linguistic families. In addition may be mentioned as especially important: Conant, L. L., *The Number Concept,* N. Y., 1896; Cushing, F. H., "Manual Concepts," in *Amer. Anthropologist,* 1892, p. 289; Eells, M., "Indians of Puget Sound: Measuring and Valuing," in *Amer. Antiquarian,* Vol. 10, p. 174; Dixon, R. B., and Kroeber, A. L., "Numeral Systems of the Languages of California," in *Amer. Anthropologist,* N.S., Vol. 9, p. 663; Powell, J. W., "Linguistic Families of America, North of Mexico," in Seventh Ann. Rpt., Bur. of Eth., 1885–86; Trumbull, J. H., "On Numerals in American Indian Languages," in *Trans. Amer. Philological Assn.,* 1874, p. 41. Full credit can scarcely be given for each statement made in this paper. The above mentioned sources have been used freely, but even more the numerous dictionaries and grammars mentioned at the beginning of this note. A bibliography of about 300 titles prepared by the author is on file in the library of the University of Chicago, the Newberry Library, Chicago, and the library of the University of Wisconsin.

[2] Major J. W. Powell: 1st Ann. Rpt. Bur. Eth., 1879–80, p. 78.

[3] For a list of these families and tribes see Powell: 7th Ann. Rpt. Bur. Eth., p. 1; Hodge: *Handbook of American Indians,* article "Linguistic Families." Not all these languages were entirely distinct, but most of them were. They are analogous to the French, Spanish, and Italian as different "languages" of the same "family." It is a conservative statement to say that about 500 distinct languages have been spoken on the American continent.

These languages differ as widely in *number words* and *number syst ms* as they do in other features. This is in marked contrast with the languages of the great Indo-European family where, even in languages which are mutually unintelligible, the same root words appear with great uniformity in the numerals. The very remarkable differences in the form and use of the numerals of the American Indians afford a fruitful field for study of the evolution of the concept of number among hundreds of distinct, uncivilized peoples. This paper is based upon an examination of the number systems of more than three hundred of these languages in North America.[4] will discuss the origin of number words and their principles of formation, the way in which they were built up into number systems, and some of the variations of these systems in actual use.[5]

I. Principles of Formation.

1. Digital Origin. The child's most natural counters are his fingers; to them he turns almost instinctively when wishing to count. What evidence is there that primitive peoples, races in the childhood state, have also turned to their digits for assistance? The answer to this question will throw much light on the origin of number words and their development into systems. We shall consider three kinds of evidence.

(*a*) *Evidence from systems used.* The almost universal prevalence of decimal, quinary, or vigesimal systems of numeration on the North American continent is perhaps the strongest general evidence that counting in its origin is digital. But the octonary, quaternary and ternary systems mentioned later will show that such evidence is neither universal nor conclusive. To this indirect evidence, based on the systems used, can be added direct proof from observation and from the ascertained meaning of number words.

(*b*) *Observational evidence.* Many observers report that Indians in various parts of the continent use their fingers, or fingers and toes, in counting, at the

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[1] *Nature of Sources.*—Before the arrival of the white man the Indian had no written language, except a system of rude hieroglyphics among some of the more intelligent tribes. Reproductions of many of these are given by Mallery in the 4th and 10th Reports of the Bur. of Eth. They have but little of mathematical interest. The oral number systems of the Indians are the important sources available. But these have been committed to writing by hundreds of different men, of varying reliability and familiarity with the languages, of different nationalities, using various systems of spelling, at different dates, and extending to various limits. Aside from this is the fact that the same Indians are often known by a dozen or twenty different names, or the same name is applied to several distinct tribes speaking different languages. Such confusion in the sources makes it extremely difficult to classify them satisfactorily for the purposes of comparative study. Many errors in detail must have been made which can only be corrected by further study and reference to experts in American linguistics in various parts of the country. But it is hoped that such errors as have been made are not serious enough to affect much, if at all, the general results and statements made in this paper.

[2] *Nature of Conclusions.*—The conclusions stated in what follows are usually based entirely on the relative *number* of languages of the 324 examined. Little effort has been made to indicate the amount of territory covered or the number of Indians involved. Some of the languages were used by only a few people, others by many thousands. But for the purposes of this paper a small tribe with peculiarities in its number system, is as interesting and as important as a much larger one.

same time speaking the corresponding number words. With some tribes the *use* of the fingers is the important thing, the accompanying vocal utterance being of secondary importance; e. g., some of the Eskimo tribes use the same *words* for 6, 7, 8, 9, 10 as for 1, 2, 3, 4, 5 but count them on the second hand. In others the language development has led to independent numerals which often preserve evidence of their digital origin. Examples are given in the next section. In widely separated tribes all over the continent actual finger counting has been observed and the rather remarkable fact noted that the order of counting is almost always uniform, commencing with the little finger of one hand and counting to the thumb, thence to the thumb of the other hand and to the little finger again. Usually the fingers are bent as the counting continues, but sometimes the hand is first clenched and the fingers then extended one at a time.[1] This general uniformity of order gives considerable aid in the linguistic discussion of the next paragraph.

(*c*) *Linguistic evidence.* The German philologist Grimm in speaking of the Old World languages says: "Alle Zahlwörter gehen aus von den Fingern der Hände."[2] Is this observation true for the New World? In the languages of civilized nations the numerals are so ground down from long usage that it is difficult to detect in their form their possible digital origin. While this is also the case in some Indian languages, in many there are striking similarities, while in others digital words and number words are almost or quite identical. Following the observed order of counting, the little fingers would be used for 1 and 10, the fourth fingers for 2 and 9, etc. Only a few typical examples of the many noted will be given with reference to each number.

The number ONE. Some of the names given the little finger are "the smallest," "the last of the hand," "little daughter of the hand." While we do not expect to find the word for *one* always or even usually connected with that for little finger (since the concept of unity doubtless preceded formal counting) yet some instances are known; e. g., Massachusetts: *pasuk* from *piasuk,* "very small"; Montagnais: *inlare,* "end is bent"; Zuni: *topinte,* "taken to start with." Two is sometimes derived from finger; e. g., Montagnais: *nake,* "another bent in"; Dakota: *nonpa,* "to bend down"; Zuni: *kwillin,* "that (finger) put down with its like." But more often it is connected with the word for "hand," probably because there are two hands.[3] THREE. The third finger is often named "the middle"; e. g., Massachusetts: *nishwe,* from *nashaue,* "half way"; Zuni: *hain,* "equally dividing one." FIVE is counted on the thumb and we have Karankawan: *natsa behema,* from *natsa,* "one," *behema,* "finger." But more prominent is the idea that the hand is completed, variously expressed as "finished," "fingers finished," "all fingers," "all done," etc.; e. g., Ojibwa: *nanan,*

[1] Is this order the natural one? Many children have been observed to use their thumb first in counting. But Cantor has noted that the South African savages use the same order as the Indian order given above. See Cantor: *Vorlesungen über Geschichte der Mathematik,* Band I, Dritte Auflage, Leipzig, 1907, pp. 6–7.

[2] Grimm: *Geschichte der deutschen Sprache,* I, p. 167.

[3] Discussed more fully below under "Duplicative Principle."

"gone," "spent," and similarly in several Eastern languages. Hidatsa: *kichu,* from *ki,* "completely," *chu,* "turned down"; Ute: *munugi,* from *manoku,* "all." In most instances however it is connected with "hand," or "whole hand"; e. g., Kaniagmiut: *talgamen,* from *talega,* "hand"; Comanche: *mowaka,* from *mowa,* "hand"; Klamath: *tunep,* from *tu,* "away," *nep,* "hand," i. e., "hand-away."

From six to nine the numerals are expressed (*a*) from the names of the fingers used; (*b*) by "hand" + 1, 2, 3, 4; (*c*) by 1, 2, 3, 4, + "again" or "besides" (most frequent); or (*d*) by 1, 2, 3, 4 repeated without change. This last method is found only among the Eskimo and is probably always accompanied by the actual use of the second hand.

SIX. The Point Barrow six well illustrates the evolution of a number word. We are given the three forms

atautyimin-akbinigin-tudlimut, literally,	"once-on next- (and) five,"
atautyimin-akbinigin	"once-on next,"
akbinigin	"on next."

Other examples of six are Tano: *manli,* from *man,* "hand," *li,* "piece," i. e., "hand and piece of next"; Klamath: *nadshk-shapta,* "one I have bent over"; Takelma: *maimis,* "finger one in." SEVEN falls on the index finger or "pointer," e. g., Zuni: *tserucek* from *tserwerc,* "to point"; Greenland: *arfinek-mardluk,* "on the other hand-two"; Omaha: *penompa,* from *pe,* "finger," *nompa,* "two." For EIGHT, Hudson's Bay: *kittukleemot,* "middle finger"; Omaha: *pethatbathi,* "finger-three"; Klamath: *ndan-kshapta,* "three I have bent over." For NINE the subtractive principle comes into use and we have the additional forms "one left," "only one," etc.[1] We also have the forms, Greenland: *mikkelerak,* "fourth finger"; Zuni: *tenalikya,* "all but one held up with rest." TEN is counted on the little finger, e. g., Hudson's Bay: *eerkitkoka,* "little finger." But more prominent is the fact "two hands completed," "man finished," or "man." Thus Zuni: *astemthla,* "all of the fingers"; Wintun: *pampa-sempta* from *pampu-ta,* "two," *sem,* "hand"; Konkau: *machoko,* from *mar,* "hand," *choko,* "double."

Above ten, various combinations of the first ten numerals occur in which of course these digital names reappear. A few other examples of interest will be given. ELEVEN. Unalit: *atkhakhtok,* "it goes down" (referring to change from hands to feet). THIRTEEN. Greenland: *arkanenpingasut,* "on the first foot, three," etc. SIXTEEN. Unalit: *gukhtok,* "it goes over" (to toes of other foot). NINETEEN. Maidu: *tsoi-ni-maiduk,* from *maidu,* "man," *tsoi,* "four"—"four with man," i. e., after 15, 4 on toward 20 (man); and similarly for 16, 17, 18. TWENTY. In decimal-system languages twenty is usually but not always "two tens." In the vigesimal it is quite commonly "man," "Indian," "all hands and feet"; e. g., Navaho: *natin,* from *tine,* "man"; Greenland: *inuk-navdlugo,* "man come to an end," or *inup-avatai-navdlugit,* "man's outer members completed"; Kaniagmiut: *swinuk,* from *suk, innuk,* "man"; Wintun: *ketet-wintun,* from *ketet,* "one,"

[1] See examples below under "Subtractive Principle."

wintun, "Indian"; Tuolomne: *renge mewoom,* "one man"; Maidu: *kom maiduk,* from *maidu,* "Indian," and *kom,* possibly "whole"; Shasta: *tsec,* from *tsec,* "man"; Tlingit: *tlekha,* from *tle,* "one," *hka,* "man."

(*d*) *Extent and distribution.* Clear linguistic digital evidence similar to the examples given above has been found in about 40 per cent. of the languages examined, uniformly distributed over the continent. Doubtless further study will reveal similar evidence in other languages especially where adequate vocabularies have not been available.

(*e*) *Non-digital evidence.* We turn now to a consideration of the evidence that the origin of number words was non-digital in some languages. There are four phases to be considered. (1) *First Four Numerals.* The concepts of unity and duality are so fundamental that in many instances we may be sure they were named before formal finger counting gave names to the corresponding words. *One* has a connection with the first personal pronoun in some languages. *Two* seems often to come from roots denoting separation, "that" as distinguished from "this," or from ideas of pairs, being frequently related to the words for hands, feet, eyes, wings, husband and wife. *Three* is more frequently digital, but it seems sometimes to have a meaning of "more," "many," a plural as distinguished from a dual. Compare Micmac: *tchicht,* "three," with the cognate Delaware *tchitch,*" still more." *Four* is sometimes expressed by a word meaning "complete," "right," "perfect." Its frequency as a sacred number among the North American Indians and its use in some cases as the base of a quaternary system[1] indicate that it is a unique word of non-digital origin. (2) *Arithmetical Operations.* Numbers higher than ten and in many cases those higher than five are expressed by arithmetical operations, and the digital meaning, even if present in the beginning, usually sinks into the background. The process of such combination begins earlier than the English in many Indian languages. We have numerous examples of $3 = 2 + 1$, $4 = 2 \times 2$, $4 = 2 + 2$, $6 = 3 \times 2$, $8 = 4 \times 2$, $10 = 5 \times 2$, $12 = 6 \times 2$, $9 = 3 \times 3$ and other rarer combinations. Thus there are many cases in which words for numerals above three are derived by purely arithmetical processes.[2] Of course there are the higher numerals, hundred, thousand, million, where they exist, in which we should rarely expect digital evidence. (3) *Marks of Completion.* When the Indian has counted ten or twenty he may use some reminder of the fact, such as a pebble, stick, arrow, grain of corn, etc. For example, Huchnom: 20, *pualya,* "one-stick-stand" and similarly for 40 and 60; in the same language 100 is *pual,* "one-stick" and similarly for 200; Maidu: 20, *penim nokom* "two arrow"; Gallinomero: 100, *tcacuto-hai,* "ten-stick." (4) *Superlative and Indefinite.* When a simple arithmetical combination is not used, especially for the expression of higher units, a superlative principle is sometimes found. Hundred is often expressed, "big ten" and thousand as "old hundred," "big hundred" or "too many to count"; e. g., Delaware: 1,000, *ngutti kittapachki,* "the great hundred"; Choctaw:

[1] See below under "Quaternary Systems."
[2] See examples below under "Duplicative Principle," "Ternary Systems."

1,000, *tahlepa siponki*, "old hundred"; Kwakiutl: 1,000,000 *tlinhi*, "number which cannot be counted"; compare the Greek "myriad."

We conclude that in North American Indian languages it is by no means true that number words, even as far as ten, always "gehen aus von den Fingern" although they probably do in a large majority of cases and the close connection can be traced in many instances. There is little uniformity as to method of formation, considerable diversity being found even in adjacent languages of the same family. This would indicate that their separation into tribes preceded the development of formal counting.

2. **Additive Principle.** Cantor says that addition and multiplication are two methods of counting as old as the formation of number words.[1] The additive principle is found of course in all numeral systems. Three phases of it are of interest in the American Indian languages.

(*a*) *Repetition.* This is the simplest form of the additive principle. If "one" is given, either as a symbol or as a word, "two" may be expressed "one-one," "three" as "one-one-one," etc., or by symbols as in the Roman numerals from one to four. In the gesture language of the Indians this is the method used, the fingers being the counters. In spoken language no instance has been found of "two" as "one-one," but there are several of "four" as "two-two"; e. g., Catawba: 2, *purra*, 4, *purrapurra*. In the Indian pictographs or hieroglyphics the simple repetition of strokes or notches is used, even for numbers up to a hundred. Sometimes these are grouped into tens by longer strokes or larger notches.

(*b*) *Addition to a Base.* The English does not begin to use the additive principle until ten is reached, but many Indian languages begin much earlier. The earliest instance found is in the Coahuiltecan: 1, *pil*, 2, *ajtic*, 3, *ajtic-pil*. In other languages we find such expressions as "6–2 added" for 8, "8 + 1" for 9, "12 + 3" for 15, and of course very often "5 + 1, 2, 3, 4," for 6, 7, 8, 9 and "10 + 1, 2, ··· 9" as in English for 11, 12 ··· 19. Those from 15 to 19 are also represented by "15 + 1, 2, 3, 4." An interesting variation is shown by the Maidu numerals from 16 to 19, which in translation are "one with man" for 16, "two with man" for 17 and similarly for 18 and 19 to 20 "man," the thought being "15 and one more on toward entire man."

(*c*) *Precedence.* Hankel and Fink call attention to a general law by which the written representation of numbers, when not confined to the mere rudiments, shows a tendency for higher numerals to precede the lower to represent addition.[2] Is there a similar tendency for the spoken order of numerals among the Indian languages? Does the lower precede the higher or vice versa? In about 250 languages sufficient facts were available for study of the method of formation of compound numerals by the additive principle. The groups from 5 to 10, from 10 to 20, and above 20 have been considered separately. Using the notation "$G + L$" to indicate "the greater is followed by the less" and "$L + G$" for the

[1] Cantor: *op. cit.*, p. 8.

[2] Hankel, H.: *Zur Geschichte der Mathematik in Alterthum und Mittelalter*, Leipzig, 1874, p. 32; Fink, K.: *History of Mathematics* (Beman and Smith trans.), Chicago, 1900, p. 8.

opposite, "$5 + 1$," "$1 + 5$," etc., to stand for pure number combinations in the order given, and "$1 + X$," "$X + 1$" for the indefinite cases of an unknown element (probably non-numerical such as "again," "besides," etc.) combined with the known numbers, we may summarize the results of an examination of these languages as follows.

In the 5–10 group for the pure number combinations, $L + G$ and $G + L$ occur with about equal frequency. But if we include the unknown compounds $X + 1$ and $1 + X$ with $G + L$ and $L + G$ respectively, $L + G$ predominates about $2 : 1$. In the 10–20 group for pure number combinations $G + L$ predominates strongly, $8 : 1$, a reversal of the order of the first group. But for $X + 1$ and $1 + X$, $L + G$ predominates slightly, the ratio being $4 : 3$, so that $G + L$ predominates in the group $2 : 1$. In the group of higher combinations $G + L$ predominates $16 : 1$. Combining these results $G + L$ predominates altogether about $2 : 1$. If the indefinite cases of $1 + X$ and $X + 1$ are excluded, only pure number combinations remaining, we approach close to a definite law. $G + L$ then predominates about $8 : 1$. $G + L$ and $L + G$ both occur in the same language in fully half the cases examined. In this particular a marked contrast with the multiplicative principle will be observed. It may be mentioned that our own (oral) system is mixed, $L + G$ from 10 to 20, and $G + L$ for higher numbers.

3. **Subtractive Principle.** Fink says: "In the verbal formation of a number system very rarely does subtraction come into use."[1] This statement is scarcely warranted for the systems of the American Indians for the principle has been found in 40 per cent. of the languages examined. It occurs most frequently in expressions for "nine"; e. g., "one finger left," "one from ten," "one from finished," etc., but also in the cases of 4, 7, 8, 14, 17, 18, 19 and the odd tens, 30, 50, etc. It is widely but not uniformly distributed over the continent It occurs most frequently in the northern, eastern and central sections of the continent, less on the Pacific coast and least of all about the Gulf of Mexico. As is to be expected, "one subtracted" is most usual, occurring in about 30 per cent. of the languages, "two subtracted" in about 5 per cent. and "three subtracted" and "ten subtracted" in about 2 per cent. each. The use of the principle is found in some languages but not in closely related ones of the same family.

(*a*) *One Subtracted.* As examples we may give for *nine*, Unalit: *keka-mitatet*, "nearly ten," or *payuk-ostau*, from *payuk*, "one," *ostau*, "less?"; Uinta: *suromatampsuin*, "near ten"; Haida: *klath-skwanson*, from *klath*, "one," and *skwansin*, "ten." For *four* we have Takhtam: *voatcham*, from *mah-atcham*, "five"; Zuni: *awiten*, "all fingers all but done." For *fourteen*, Point Barrow: *akimiaxotaityuna*, "I have not quite 15." For *nineteen*, Alaska: *enuenok-otalia*, "twenty less one." *Thirty-nine*, Kulanapo: *pitikunanu-akhokaki*, "forty, one not." Arikara expresses both 7 and 9 from 8 and 10 respectively by means of a diminutive particle, "little ten," "little eight."

[1] Fink: *op. cit.*, p. 8.

(*b*) *Two Subtracted.* Onondaga: 8, *teg-ueron,* from *tegni,* "2"; Crow: 8, *nupa-pik,* from *nupa,* "2," and *pirake,* "10"; Kwakiutl: *matl-gwanatl,* from *matl,* "2."

(*c*) *Three Subtracted.* This occurs but rarely. Pawnee is noteworthy. In it, 17, 18, 19 are *tauit-kaki, pitkus-kaki, usku-kaki*. from *usku,* "1," *pitkus,* "2," *tauit,* "3," and *kaki,* "less."

(*d*) *Ten Subtracted.* This is quite common among some of the California families, where the odd tens, 30, 50, 70, 90 are expressed as 40, 60, 80, 100 combined with ten, with the thought "60 lacking 10." In some cases however the thought may be "(40) and 10 on the way to 60," a variation of the additive principle not uncommon among these tribes [1]

4. **Multiplicative Principle.** This principle, like the additive, is of universal use in Indian as well as in all other languages for the formation of higher numerals. It seems to begin in the Indian languages with the expression of 4 as "2×2." It is constantly used in the formation of "secondary bases" in the ternary, quaternary, quinary, decimal, and vigesimal systems. Our chief interest in studying this principle is in the question of precedence. A much more decisive law of precedence than in the case of the additive principle is found. About 200 languages afford data for the conclusion that the marked tendency in the formation of higher numerals by multiplication is for the lesser number to precede the greater. This coincides with both the oral and written English forms. In every group the type $L \times G$ predominates over the type $G \times L$ to a marked degree. For the formation of the tens, forms of the type 2×10 predominate over the type 10×2 in the ratio 5 : 1; for the hundreds, 2 : 1; for the thousands, 2 : 1; for others (e. g., $8 = 2 \times 4$), 17 : 1. Or altogether a predominance of $L \times G$ 4 : 1. Almost all the instances of $G \times L$ are found in the languages about the Gulf of Mexico and in the single but large Siouan family of the plains. If the Siouan languages alone were left out of consideration the predominance of $L \times G$ would be about 8 : 1. In only five languages do we find $L \times G$ and $G \times L$ both occurring in the same system, a decided contrast to the results noted in the study of the additive principle.

5. **Duplicative Principle.** A striking feature of the Indian numeral systems is the frequency of occurrence of a duplicative or pairing principle. In some instances 6 is expressed as "2×3," "again 3," "3, 3," "threes" and similarly for 4, 8, 10, and even 12. The large number of natural pairs, such as the eyes, hands, arms, wings, etc., suggests that counting by pairs might be the course of evolution followed by some languages. The standard historians of mathematics make little reference to such a principle as of any importance in the numeral systems of primitive people.[2] We do find Hankel, however, making the rather surprising statement in discussing the number words of uncivilized people, that ten is never expressed as two times five, but always by a simple number word.[3] This is not the case in American Indian languages; e. g., Gabrieleno: *wehes-mahar,* from *wehe,* "2," and *mahar,* "5"; Serranos: *wor-maharte,* from *maharte,* "5,"

[1] See example of Maidu given under "Additive Principle."

[2] Cantor: *op. cit.*, p. 5. [3] Hankel: *op. cit.*, p. 20.

wor, "2"; Patwin: *pampa-semta*, from *eti-semta*, "5," *pampet*, "2." Examples of its use for 4, 6, 8, 12 are Kutchin: 6, *neckh-kiethei*, 8, *nakhei-etanna*, from *nackhai*, "2," *kiethai*, "3," *etanna*, "4"; Kansas: 8, *kiya-tuba*, from *tuba*, "4," *kiya*, "again"; Shoshone: 4, *what-sowit*, from *what*, "2"; Cehiga: 12, *cape-nanba*, from *cape*, "6," *nanba*, "2"; Chwachamaju: 8, *kom-tca*, from *mitca*, "4," *ko*, "2."

In languages of the same family and even dialects of the same language there is variation in the use of this principle. And many examples can be given where the principle is not used at all in languages closely related to those in which it is. One feature is rather surprising; namely, of the approximately 125 cases of the probable use of this principle, it is far more common in the formation of four, six, eight than in the case of ten, even though ten is represented so commonly by the two hands. Fifty instances of its use for the formation of "8" are found, thirty-five for "4" (although some of these are somewhat obscure), twenty-five for "6," only ten for "10" and two for "12." Several languages seem to use it regularly in the formation of the even numbers to ten.

6. **Divisive Principle.** Historians of mathematics agree on the rarity of the use of this principle in the formation of numerals the world over.[1] Only two or three possible instances have been found in Indian languages. Thus we have Unalit: 10, *kolin*, and the literal meaning of the word, "upper half of the body," Point Barrow: 10, *kodlin*, "upper part (of body)," and similarly in other Eskimo languages. One other example has been found, Pawnee: 5, *sihuks*, "hands half" from *ishu*, "hand," and *huks*, "half," i. e., "half of the two hands."

7. **Fractions.** Although many tribes had numeral systems of integers running into the thousands and even to millions,[2] very few of them had any idea of fractions. Where we do find such ideas they are of the most rudimentary sort. "One half" occurs most frequently, but only in about a dozen cases as far as noted, while examples of the other fractions are almost negligible.[3] It is worth mentioning that the few instances we find are all of "unit fractions."[4] Onondaga shows the best development of fractions, but how meager for a language whose numerals are given to one million. Its fractions are: $\frac{1}{2}$, *sat wachenonk*, meaning uncertain; $\frac{1}{3}$, *achen-na-degayagui*, from *achen*, "3," meaning "thrice divided"; $\frac{1}{4}$, *gayeri-degayagui*, from *gayeri*, "4," "four times divided."

8. **Notation.** As already mentioned, few symbols for number other than the spoken words are found. But on grave posts, buffalo robes, tattooing and other mnenomic pictographs there are a few pictured symbols.[5] The usual method used for indicating numbers is by the repetition of single strokes, i. e., the additive

[1] Fink: *op. cit.*, p. 8; Hankel: *op. cit.*, p. 21.

[2] As to actual use of these numbers, see below, "Limits in Use."

[3] Some ethnologists state that words for the simpler fractions were common, but that they have been recorded but rarely by students of Indian languages.

[4] The pedagogical suggestion is worth noting. The use of integers developed long before fractions, and when fractions are introduced the unit fractions, i. e., those which have unity for the numerator, are the first to appear. In the Ahmes papyrus, one of the earliest known mathematical documents (Egyptian), a table is given for reducing common fractions to unit fractions, which were regarded as the standard type. See Cajori: *History of Mathematics*, New York, 1894, p. 14; Ball: *Short History of Mathematics*, London, 1893, p. 3; Cantor, *op. cit.*, Chap. I.

[5] For reproductions of all those known in North America and discussion of them see articles by Mallery in 4th and 10th Ann. Rpts. Bur. Eth., Washington.

principle in its simplest form. Sometimes the strokes are arranged in rows of ten or every tenth stroke is made longer than the others. Instances of this kind are found for the expression of numbers as high as thirty. They are found among the more highly civilized Indians of the middle west, especially among the Dakotas. On the other hand the Comanches are said to have been "ignorant of the elements of figures, even of a perpendicular stroke for one."[1] Pure numeral notation is not always found. Frequently the number of objects is expressed by the repetition of the symbol for the object the desired number of times, especially in the case of men or tepees. Another method is by dots. A man's head with eight dots above it in one case means nine men, the head itself counting for one. Another picture gives a head over which are thirty black dots in three lines of ten each. This is said to mean thirty men, not thirty-one. Thus the usage is not fixed. The use of notches cut on sticks was frequent, not only in the middle west but in California and on Puget Sound in western Washington. At the San Gabriel mission in California every tenth notch was cut entirely across the stick instead of only in the corner.

(*To be continued.*)

SYNTHETIC PROJECTIVE GEOMETRY AS AN UNDERGRADUATE STUDY.

By W. H. BUSSEY, University of Minnesota.

Modern Geometry.—Synthetic projective geometry is called modern synthetic geometry because the theories and methods that make of its propositions a homogeneous and harmonious whole are due for the most part to men of a time near our own, namely Poncelet, Möbius, Steiner, Chasles, Von Staudt, Reye, Cremona, and others. Some of its theorems can be traced back to Euclid, Apollonius, and Pappus, but it is in the work of Pascal and Desargues that the germs of the modern theories are to be found. When he was only sixteen years old Pascal discovered his famous theorem about a hexagon inscribed in a conic and made it the basis of a treatise on conics which is unfortunately lost. It was never published but we know something of its contents and extent from a report made by Leibnitz who saw it in Paris. Two of the most important theorems of projective geometry go by the name of Desargues. One is that if any two triangles in a plane or in space are such that corresponding vertices are collinear in pairs with a fixed point, then the corresponding sides are concurrent in pairs with a fixed straight line. The other is that any transversal meets a conic and the pairs of opposite sides of any inscribed quadrangle in points of an involution. But unfortunately for the development of the subject the work of Pascal and Desargues was overshadowed by the geometry of their contemporary Descartes. Analytic geometry and the calculus, invented about thirty years later, became the

[1] Eakins, D. W.: in Schoolcraft's *Indian Tribes*, Philadelphia, 1851–60, Vol. 1, p. 416.

chief instruments of progress in geometry as in all mathematics, and the work of Pascal and Desargues lay undeveloped for nearly 200 years. The revival of synthetic geometry by methods similar to theirs was due chiefly to Poncelet, and for that reason the origin of projective geometry is usually ascribed to him.

Synthetic versus Analytic Geometry.—Synthetic projective geometry is still overshadowed by analytic geometry and the calculus and it does not have the place in the undergraduate curriculum that it ought to have. Indeed it may be said that it has no recognized place, for in most American colleges and universities it is not offered at all, and there is nothing like uniformity in the conditions under which it is offered in those that do give a course in the subject. In some colleges it is primarily for graduates and in others it is primarily for undergraduates. More often it is for both undergraduates and graduates. The subject is offered three hours a week for a semester, two hours a week for a year, and three hours a week for a year. A number of colleges are offering an introductory course as an elective to juniors and seniors who have had a first course in the calculus. In a few cases the only prerequisite is a first course in analytic geometry and the course is open to sophomores. When given under these conditions the course must of necessity be very different from that given in some institutions to graduate students who have a reading knowledge of French and German. At Bryn Mawr a lecture course in the subject is offered as a post-major course to juniors and seniors who have taken college mathematics five times a week for two years, and a reading knowledge of French and German is a prerequisite.

The subject if properly taught is well within the grasp of the undergraduate who has had a first course in the calculus. An introductory course of at least three hours a week for a semester should be offered to such students in every college, and those preparing to teach mathematics in the high school should be encouraged to take it. The subject is so recommended at Vassar, the prerequisite being analytic geometry. It is a significant fact that most of the prominent colleges for women offer courses in synthetic projective geometry to junior and senior undergraduates. Bryn Mawr, Mount Holyoke, Smith, Vassar, Wellesley, Wells, and Goucher College of Baltimore do so. The course at Bryn Mawr requires considerable mathematical maturity and a reading knowledge of French and German, but at the other colleges the course is less advanced. At two of them the prerequisite is analytic geometry.

Curriculum Now Planned for the Prospective Graduate Student.—The reason why projective geometry has no well-defined place in the undergraduate curriculum is not hard to find. The curriculum is planned primarily for those who are to go on with graduate work in mathematics or physics. It takes two years, sometimes two and a half, for the student to get through college algebra, trigonometry, analytic geometry, and a year of the calculus. Then if he is to be well prepared for graduate work he ought to take advanced calculus, solid analytic geometry, and a course in determinants and the theory of equations; a second course in plane analytic geometry is needed for a working mastery of the subject,

and he ought to have at least two years of undergraduate physics and one of astronomy, and perhaps a year of theoretical mechanics. He is not likely to have time for any more electives. In fact most graduate students have not had as good a preparation as this. One or more of these subjects is likely to be left for the first graduate year. But every one of the subjects is of more importance to the prospective graduate student than synthetic projective geometry, because they are all on the main line of progress in mathematics and are all prerequisites for advanced work in applied mathematics. Projective geometry is more an end in itself. It is not needed so much as a prerequisite for something else. It is not a prerequisite for graduate work in mathematical physics or astronomy. So it is left out by most prospective graduate students even if they have an opportunity to get it as undergraduates, as most of them do not. They get it as graduate students or they do not get it at all. Then they are more mature and the course can be made, and usually is made, to cover much more ground and to be more intensive than an undergraduate introductory course in the subject. A reading knowledge of French and German can be assumed, and there need be no hesitancy in making the course one in pure geometry. It can be as abstract as the teacher pleases and it ought to involve to a greater or less extent a study of the foundations of geometry. Properly qualified seniors could take such a course, but it is not suitable for those who have just completed a first course in the calculus.

Part of a Liberal Education.—The students who take elective courses in mathematics after having had a first course in the calculus are of three kinds; those who are preparing to do graduate work in mathematics, physics, or astronomy; those who are going to college for a liberal education and who are taking mathematics because they like it and not because they expect to use it; and those who are taking mathematics in college as a preparation for teaching mathematics in the high school. For reasons already given a student of the first kind can afford to wait and take projective geometry when he is a graduate. If he is interested primarily in physics or astronomy, he may not want it at all. A student of the second kind is prepared by his one year's work in the calculus for the undergraduate courses in physics and astronomy that he may wish to take. Any other courses in mathematics that he elects he will choose on account of their interest to him. He may be one of those who always liked geometry better than algebra, and there are many such. If he belongs to this class, a course in modern synthetic geometry would probably be more interesting to him than any other course that could be offered. At any rate he ought to have a chance to take it if he wants to.

Preparation for Teaching.—These remarks about the student who is taking mathematics as a part of a liberal education apply equally well to the student of the third kind who is taking mathematics as a preparation for teaching. In many institutions this class is the largest of the three. In coeducational colleges it consists mostly of young women. College algebra, trigonometry, analytic geometry, and the calculus are a better preparation for teaching algebra than

they are for teaching geometry. They involve almost constant algebraic manipulation even when their subject matter is not algebra. It is often said that a student is not really a master of high school algebra until he has been through a course in the calculus. Many prospective high school teachers take, in addition to these subjects, a course in determinants and the theory of equations and thereby make their preparation for teaching algebra so much the better. But what preparation for teaching geometry does the prospective high school teacher get that is at all comparable with the preparation for teaching algebra afforded by these courses? Of course all courses in mathematics are indirectly good preparation for teaching geometry. Experience in one branch of mathematics counts in the study of any other branch to some extent, but that does not alter the fact that, as it now stands, the undergraduate curriculum in mathematics affords a better preparation for teaching algebra than geometry. Algebra and analysis have the right of way. The geometry of the high school is synthetic. It consists largely of memory work and deductive reasoning. It holds its place in the high school not only because it is a useful subject in the work of the world but also because it is a good course in logical reasoning. What better preparation for teaching it can the student get than a course in synthetic projective geometry? It is more closely related to the geometry of Euclid than any other branch of mathematics. It gets the student back to synthetic methods. It involves more of reasoning and less of the technique of manipulation than either analytic geometry or the calculus, and it can be taught so as to be as easily within the mental grasp of the college sophomore as either of these subjects. Construction problems with the ruler and compass, with the ruler alone, with the graded ruler, and with the parallel ruler should be an important part of the introductory undergraduate course in synthetic projective geometry. There is a wealth of such problems in Cremona's *Elements of Projective Geometry*, and the teacher can find more, if they are wanted, in Adler's *Geometrische Konstruktionen*. To draw a line through the inaccessible point of intersection of two given lines by means of the ruler alone, to draw a line through a given point and parallel to a given line by means of the graded ruler or the parallel ruler, and the many other constructions possible by the principles of projective geometry are fascinating to lovers of geometry, and the constructions that can be made by means of the theorems of Pascal, Brianchon, and Desargues are a revelation to the student. What better preparation is there than this for teaching that part of the high school course which deals with construction problems?

Generality and Power of Synthetic Geometry.—Many a student leaves college to become a teacher of high school geometry with the notion that no progress in geometry is possible except by means of coördinates and algebra, and that there is no higher geometry more closely related to the geometry of Euclid. This ought not to be so. The course in modern synthetic geometry would not be so interesting to the student nor so valuable in broadening his mental horizon if it were merely a sequel to Euclid. It is more than that, inasmuch as it is characterized by the great generality and power of its methods and theorems. It is

said that Pascal, in his lost treatise on conics, had 400 corollaries to his famous theorem of the mystic hexagram. The method of projection whereby theorems about circles are translated into theorems about conics; the principle of duality whereby every theorem about the incidence of points, lines, and planes yields another about planes, lines, and points; and the principle of continuity, which Chasles prefers to call the principle of contingent relations,—these are very general in their applications and are of great power in research. There is nothing to match them in the geometry of the Greeks. Their geometry consists for the most part of isolated theorems. They were more anxious to convince than to explain, and the student gets no information about the way in which the various theorems were discovered. In modern synthetic geometry, on the other hand, the student is to a considerable extent led to understand how many of the theorems came to be discovered. A good example of the great power of the principle of continuity is afforded by the long list of deductions that can be made from the theorems of Pascal and Brianchon. The student can discover some of them for himself as soon as he is let into the secret of the method. The principle of continuity indicates what the theorems are. Afterwards they can be proved rigorously. This modern synthetic geometry is very different from the synthetic geometry of the Greeks both in subject matter and in method, but it has enough in common with it to make it a good preparation for the teaching of high school geometry.

One Semester for the First Course.—The college student who is preparing to be a high school teacher must be qualified to teach more than one subject, at least in the beginning of his teaching career. Teachers in small high schools sometimes have to teach four or five. Certain courses in psychology and education are required for state teachers' certificates, and it is expected that every high school teacher will have a good liberal education. The student who is going to be a teacher with mathematics as a specialty ought to take courses in college physics and astronomy. For these reasons the prospective teacher often finds that he cannot take more than one or two half year courses in mathematics after he has had a year of the calculus. In many colleges the second half year of the calculus is normally taken in the first half of the student's junior year. Synthetic projective geometry ought to be available for these people, many of whom will be good high school teachers of geometry although they would not all, or even nearly all, be successful in graduate work in mathematics. But the course should not be so arranged that the student must take a whole year of the subject. It should be a half year course, or if a year's work is offered it should be so arranged that the work of the first semester constitutes a good course by itself, just as the first course in analytic geometry is one that is worth while even if the student never takes any more of it. A good introduction to projective geometry can be given in a half year, as in the case of analytic geometry. Then the second half year's work can be thought of as a second course for those who want it and have time to take it. It will be on a par with the second course in analytic geometry as far as its relation to the first course is concerned. In some

colleges a half year of synthetic projective geometry is followed by a half year of advanced analytic geometry, or vice-versa, neither being a prerequisite for the other. Sometimes the two are offered in alternate years so that the student who wishes can get either or both. This plan is necessary in some colleges because there is not enough demand for both courses every year. Another plan is to give a year's course in modern geometry, synthetic and analytic methods both being used all the way through the course. This makes a good year's course but is not so well adapted to the needs of the student who can take only one semester's work. This plan is followed in a number of colleges, but many times the course announced as modern geometry, whether it is a year or a half year course, is too exclusively taken up with the analytic point of view at the expense of synthetic projective geometry. This is due to some extent to the fact that many college teachers of mathematics have never had an adequate training in modern synthetic geometry, and consequently their interest is on the analytic side. This is another consequence of the fact that projective geometry is not on the main line of progress in mathematics. Thus the present status of modern synthetic geometry tends to perpetuate itself.

A Brief Outline of a Suitable Course.—The introductory course of a half year three times a week, or its equivalent, should not be a course in pure geometry that avoids the measurement of geometric magnitudes and all metric properties, like the geometry of Von Staudt. Nor should it be a study of the foundations of geometry, or involve such a study, to any considerable extent. The plane geometry of the high school should be assumed in its entirety just as it is for the first course in analytic geometry. The course will from its very nature deal largely with projective rather than metric properties of geometrical figures, but to avoid all metric notions is not wise in a course open to college juniors. Anharmonic ratios should be used freely, and the measurement of geometric magnitudes is involved in their definition.

The following is a brief outline of a course that can be completed in a half year, if not too much time is given to any one of the topics. The author of this paper has tried it with satisfactory results:

Central projection and figures in perspective, points at infinity, the principle of duality, projective geometric forms, harmonic forms, anharmonic ratios, the construction of geometric forms, projective forms in relation to the circle, projective forms in relation to the conic sections, the theorems of Pascal and Brianchon and their many corollaries, the principle of continuity, projective ranges on a conic and projective series of tangents to a conic, superposed projective ranges and involutions, the theorem of Desargues about a conic and an inscribed quadrangle, problems of the second degree, pole and polar, centers and diameters. Construction problems to be given all the way through the course, as many as there is time for.

The Question of a Suitable Text.—There is no satisfactory textbook for this course. Cremona's *Elements of Projective Geometry* can be used to advantage if it is supplemented to a considerable extent by lectures, and if good judgment is used in omitting some paragraphs and in substituting for others. The writer has used the book in this way but has found it necessary at times to substitute other proofs for the ones given by Cremona because his proofs involved para-

graphs that had been omitted. Or the course can be made a lecture course with considerable work in assigned problems and outside reading. Reye's *Geometry of Position* (volume 1, English translation by T. F. Holgate) is out of print, but it is likely to be in the library so that it can be used for reference. The *Projective Geometry* by Veblen and Young is too advanced and abstract for students who have had only two years of college mathematics, and besides it is not suitable for a short course. But it can be used to advantage for reference at times. Emch's *Projective Geometry* and Miss Scott's *Modern Geometry* get at the subject mostly from the side of analytic geometry, but they can be used for reference at times. There are some English books that the teacher may want to have in the library for occasional reference, although no one of them is suitable for use as a textbook for the course, namely, Russell's *Elementary Treatise on Pure Geometry*, Milne's *Elementary Treatise on Cross Ratio Geometry*, Filon's *Projective Geometry*, Pickford's *Elements of Projective Geometry*, and Durell's *Plane Geometry for Advanced Students*.

It is to be hoped that some one will publish a book on synthetic projective geometry that will be suitable for use as a textbook for an introductory course in the subject. In the meantime the course if given will have to be conducted in one of the ways suggested above.

A NOTE ON THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS.

By C. R. MAC INNES, Princeton University.

In showing how to solve linear differential equations with constant coefficients, the usual way of finding the complementary function seems to be to guess at the various parts and then to prove that the guesses are correct. A much more satisfactory way is to define the differential operators $D - \alpha$, $D - \beta$, etc., to show that they obey the laws of addition and multiplication, and then to proceed in a straightforward way to solve the problem.

Having shown that $\frac{d^2y}{dx^2} - (\alpha + \beta)\frac{dy}{dx} + \alpha\beta y = 0$ may be written as $(D - \alpha)(D - \beta)y = 0$, one sees that a solution of $(D - \beta)y = 0$ is a solution of the original equation since $(D - \alpha)0 \equiv 0$. Since the factors are commutative any one may come last and we can get all the parts of the complementary function except those arising from repeated factors.

If $(D - \alpha)(D - \beta)y = X$, put $(D - \beta)y = v$; then $(D - \alpha)v = X$. Integrating this in the usual way one gets

$$v = (D - \beta)y = e^{\alpha x}\int e^{-\alpha x}Xdx.$$

Hence the rule: *To remove a factor $(D - \alpha)$ from the left side, multiply the right member by $e^{-\alpha x}dx$, integrate and multiply by $e^{\alpha x}$.*

If we have $(D - \alpha)^2 y = 0$, we may therefore get $(D - \alpha)y = c_1 e^{\alpha x}$. On repeating this operation,

$$y = e^{\alpha x} \int e^{-\alpha x} c_1 e^{\alpha x} dx = c_1 x e^{\alpha x} + c_0 e^{\alpha x}.$$

If we have $(D - \alpha)^3 y = 0$, we get in precisely the same way

$$y = c_2 x^2 e^{\alpha x} + c_1 x e^{\alpha x} + c_0 e^{\alpha x}.$$

The complementary function would then be completely known on proving the usual theorem that if $v_1, v_2, \cdots v_n$ are solutions of $f(D)y = 0$ then $c_1 v_1 + c_2 v_2 + \cdots c_n v_n$ is also a solution.

One need hardly draw attention to the fact that the complete value of y will be obtained if the factors of

$$(D - \alpha_1)(D - \alpha_2) \cdots (D - \alpha_n) y = X$$

are removed in turn by the above rule, the constant of integration being introduced each time.

A GRAPHICAL SOLUTION OF THE DIFFERENTIAL EQUATION OF THE FIRST ORDER.*

By T. R. RUNNING, University of Michigan.

The following graphical solution of the differential equation of the first order was derived while trying to solve a problem involving two such equations. As an illustration, these equations will be solved after a discussion of the method of solution has been given.

If the primitive of a differential equation which expresses the condition of a problem can not be found, or can be found with difficulty, it may be desirable to have a method of approximate solution which does not involve a great deal of labor. It is thought that the method given is more easily carried out than any other which has come to the writer's attention.[1]

The solution depends upon the simple fact that the area under the derived curve is equal to the corresponding ordinate of the integral curve. In Fig. 1

$$\text{Area } OPQ = RQ.$$

If in the equation

$$\frac{dy}{dx} = f(x, y)$$

we assign particular values to y and plot the resulting equations with dy/dx as

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* First part of a paper presented to the American Mathematical Society at Madison, September 9, 1913.

[1] For a comparison of approximate methods see "Beitrag zur näherungsweisen Integration totaler Differentialgleichungen," by W. Kutta, *Zeitschrift für Mathematik und Physik*, Vol. 46, pp. 435–453.

ordinate and x as abscissa, we shall have, designating each curve by the corresponding value of y, a family of curves as shown in Fig. 2.

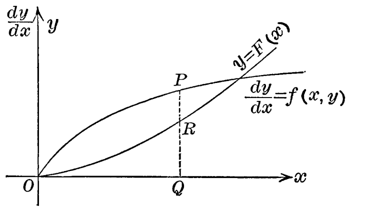

FIG. 1.

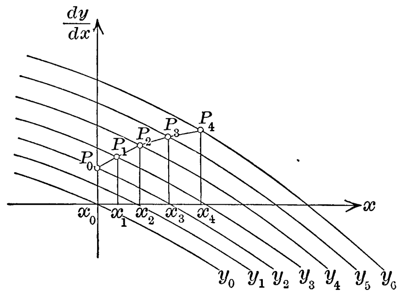

FIG. 2.

Suppose the initial condition to be $y = y_2$ when $x = 0$. Draw a curve, which for an approximation may be a straight line, from P_0, where the curve y_2 crosses the axis of dy/dx, to the curve y_3, such that the area under it shall be equal to $y_3 - y_2$. From this point, P_1, draw a line to a point P_2 on the curve y_4 such that the area under the line P_1P_2 shall be equal to $y_4 - y_3$. This process continued will give any desired number of points on the integral curve. The curve connecting the points $P_0P_1 \cdots P_n$ is the derived curve.

It is to be noted that the integral curve is not obtained by graphically integrating the derived curve, but the coördinates of its points are read from the figure. Thus (x_0, y_2), (x_1, y_3), (x_2, y_4), (x_3, y_5), etc., are points which determine the integral curve. If a close approximation is required the values of y must be so chosen that the corresponding curves will lie near together.

The problem mentioned was one which gave rise to the two differential equations

$$\frac{dy}{dx} = \frac{1800x - 132y^{\frac{3}{2}}}{11y^{\frac{6}{5}} + 1000}, \tag{1}$$

$$\frac{dy}{dx} = \frac{1740 - 240x - 132y^{\frac{3}{2}}}{11y^{\frac{6}{5}} + 1000}. \tag{2}$$

It was required to find the maximum value of y and the corresponding value of x, knowing that when $x = 0$, $y = 0$, equation (1) was to be satisfied between $x = 0$ and $x = \frac{29}{34}$, equation (2) between $x = \frac{29}{34}$ and $x = 7\frac{1}{4}$. There was another equation to be satisfied beyond $x = 7\frac{1}{4}$ but, as the conditions clearly showed, the maximum value of y occurs before x reaches the value $7\frac{1}{4}$, it need not be considered.

The curves which correspond to different values of y are straight lines and shown in Fig. 3.

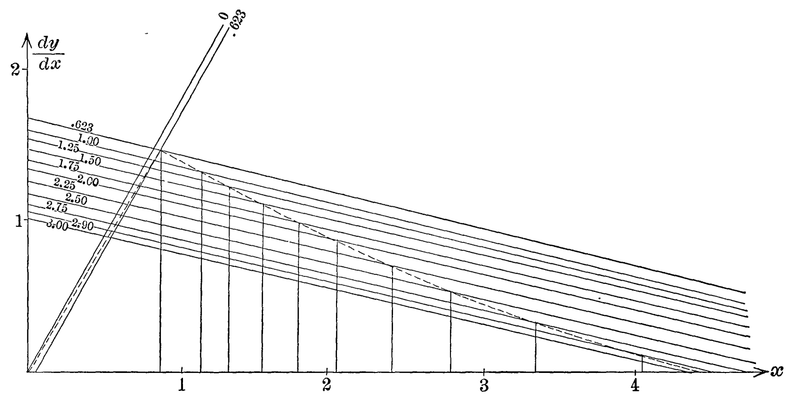

Fig. 3.

y will be a maximum when $dy/dx = 0$. This value is 2.926 and the corresponding value of x is 4.492. The values chosen for y are indicated on the figure. It is seen that when $x = \frac{29}{34}$, $y = .623$. This value must, therefore, be used in both differential equations.

The method was applied to the problem given in Kutta's article, above referred to, for comparing the five different approximations. With squared paper ruled to twentieths of an inch and using two inches as the unit, the approximation was closer than that of any one of the first four but not quite so close as Kutta's, while the work necessary to obtain the result was much less.

In a later paper it is proposed to extend the method to the solution of the differential equation of the second order.

BOOK REVIEWS.

W. H. Bussey, Chairman of the Committee.

Practical Mathematics. By Claude Irwin Palmer, Assistant Professor of Mathematics in the Armour Institute of Technology. The McGraw-Hill Book Company, New York, 1913. Four volumes. \$0.75 each.

We have here in four small, neatly bound and clean cut volumes a very interesting presentation of elementary mathematical subjects from Arithmetic through Trigonometry. The work has been prepared for persons of maturity, being, as stated by the author, the outgrowth of a course in practical mathematics given in the evening classes of the Armour Institute of Technology.

The titles of the four little volumes are:

Part I. Arithmetic with Applications (150 pages).
Part II. Geometry with Applications (166 pages).
Part III. Algebra with Applications (189 pages).

Part IV. Trigonometry and Logarithms (160 pages).

These volumes are admirably adapted to meet the needs of the persons to whom they are addressed. The treatment throughout is simple and concise; demonstrations are largely avoided; and the whole work abounds in interesting practical exercises drawn from a large variety of sources. These exercises serve to make the books of great value to all teachers of elementary mathematics as a source of fresh and instructive problems.

W. C. BRENKE.

Computing Tables and Mathematical Formulas, arranged for the use of high schools and colleges. By E. H. BARKER. Ginn and Company, Boston, 1913. Narrow 12 mo, semi-flexible cloth, v+88 pages. $0.75.

This admirable little book of tables, 4 inches by 7 inches, contains a wealth of information which furnishes material for many problems of ordinary computation besides being very convenient for problems involving logarithms. Fifteen tables present the ordinary trigonometric functions and logarithms as well as powers (squares and cubes), roots, circumferences, areas, for the numbers from 1 to 1,000, trigonometric formulas, volumes of spheres (1 to 100 in diameter), standard gauges, decimal equivalents of 32ds and 64ths, specific gravities, weight of a cubic foot of various materials, and convenient equivalents. Any high school pupil, and even the ordinary college student, would profit by having such a book placed in his hands. The objection to the work from this point of view, however, is the price. Compare this book of 88 pages at 75 cents with the *Vierstellige Tafeln* by Schubert-Haussner at 22 cents. The German work is in two colors and consists of 175 pages, being volume 81 of the well-known *Sammlung Göschen*, G. J. Göschen'sche Verlagshandlung, Berlin W. 10, Genthiner-Strasse 38. Under the new tariff law teachers will be able to import such works free of duty for their students and until similar tables are published at a more moderate price in this country the German tables are to be preferred. Aside from the price, even, it is highly desirable that our high school students should occasionally see and handle foreign books.

L. C. KARPINSKI.

An Introduction to Mathematics. By A. N. WHITEHEAD. Henry Holt and Co., New York, 1911. 256 pages. $0.75.

Working mathematicians have usually not taken the time and trouble to render their point of view, the nature of their subjects of interest, and their methods intelligible to a layman. This state of things has no doubt been brought about by a feeling widely prevalent that the layman, even though intelligent, is not able to understand these things without first passing through a considerable course of training. So far as results and details of work are concerned this feeling is certainly justified.

But it is at least open to question whether the really fundamental ideas, stripped of all technicality in verbiage and results, are not after all of such simple

character as to be capable of presentation to any intelligent reader who may be interested in knowing what they are and how they are related to life and thought.

If such a presentation could be made it would obviously be of no little advantage to the science of mathematics. Many of us must have felt our extreme isolation from our fellow-workers in other fields; and we would welcome anything which would properly serve to narrow the gap between us. One great means of scientific progress is in the interconnections and interrelations between separated domains of thought.

Again, all of us have noticed how often the study of mathematics in the college (and even in the high school) begins with disappointment. The ideas and processes to which the student is early introduced do not command his interest. May it not be true that we give too much attention to the trivialities of computation and too little to the fundamental principles, always interesting, on which the science is built?

It appears, then, that both the student and the general reader might find it of great value to have at hand a simple and straightforward, yet accurate, discussion of the fundamental ideas of the mathematical sciences. This is what the author of the little volume at present under review has given us in a lucid and interesting exposition. It augurs well for a dissemination of interest in mathematical questions when men of such ability as Dr. Whitehead are willing to take the time to induct others into a clear apprehension of the elementary and fundamental notions of the science.

The object of this book is not to teach mathematics, but to enable students from the very beginning of their course to know what the science is about, and why it is necessarily the foundation of exact thought as applied to natural phenomena. There is no technical mathematical detail to embarrass the non-mathematical reader; and yet even the student of mathematics will find in it many illuminating statements. The book deserves a wide circulation.

It can be recommended unhesitatingly to any layman who desires to know the simple conceptions which lie at the basis of modern mathematics and to ascertain how they have grown up during the progress of thought.

A general idea of the contents of the book may be gained from the following list of chapter headings: the abstract nature of mathematics, variables, methods of application, dynamics, the symbolism of mathematics, generalizations of number, imaginary numbers, coördinate geometry, conic sections, functions, periodicity in nature, trigonometry, series, the differential calculus, geometry, quantity.

R. D. Carmichael.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

ALGEBRA.

396. Proposed by H. E. TREFETHEN, Colby College.

Show that $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots = \sqrt{2}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$.

GEOMETRY.

424. Proposed by H. E. TREFETHEN, Colby College.

In a given triangle ABC, determine by geometric demonstration the point O so that the sum of the distances, $AO + BO + CO$, shall be a minimum.

425. Proposed by V. M. SPUNAR, Chicago, Ill.

Find the ratio of the areas, A_1 and A_2, of the parabolas formed by projectiles whose ranges are the same and whose angles of projection are complements of each other.

CALCULUS.

345. Proposed by C. N. SCHMALL, New York City.

"Of all the quadrilaterals which can be formed from four given sides, that which is inscriptible in a circle has the greatest area."

[From GOURSAT-HEDRICK, *Math. Anal.*, p. 133, ex. 5.]

Proposer's Remark.—Prove this, and furthermore show that when only three sides are known, the length of the fourth side of the maximum quadrilateral is the root of a cubic equation.

346. Proposed by C. N. SCHMALL, New York City.

Given the height of an inclined plane, to find its length so that a given force, acting on a given mass in a direction parallel to the plane, may draw it up in the shortest time.

Note.—This question has a practical application in the case of a truck, of known height, upon which a mass is to be loaded by means of skids.

MECHANICS.

280. Proposed by C. N. SCHMALL, New York City.

Given the distance d between two smooth hooks in the same horizontal line. Show that the shortest string which can form a catenary, with these hooks for points of support, is de, where e is the base of the Naperian system of logarithms.

281. Proposed by C. N. SCHMALL, New York City.

ABC is a triangle inscribed in a circle, center O, and L, M, N, are the centers of gravity of the sectors AOB, BOC, COA. Show that

$$\frac{AB}{OL} + \frac{BC}{OM} + \frac{CA}{ON} = 3\pi.$$

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

197. Proposed by E. T. BELL, Seattle, Wash.

Show that in the expansion of $\dfrac{1 + z + z^2 + \cdots + z^{p-1}}{(1 - z)^{p-1}} - 1$, where p is a prime, the coefficients of the various powers of z are divisible by p. [EISENSTEIN, *Crelle*, *t. 27, p. 282.*]

198. Proposed by ARTEMAS MARTIN, Washington, D. C.

Prove that every even number is the sum of two prime numbers.

Note.—This problem has long been known and no proof has ever been given. [EDITORS.]

CORRECTION.—Number **192**, incorrectly given as **188** in the June issue, page 196, should have the zeros on the right of equations (1), (2), (3) each replaced by □, the symbol for a "square number."

SOLUTIONS OF PROBLEMS.

ALGEBRA.

386. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Given the sequence, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{1}{6}$, $\frac{5}{6}$, $\frac{1}{7}$, $\cdots$, u_1, u_2, u_3, $\cdots$, show that

$$\left.\frac{\sum_1^n u_i}{n}\right]_{n\doteq\infty} = \frac{1}{2}.$$

SOLUTION BY H. A. LEVY, Houghton, Mich.

Consider the sequence $\frac{1}{q}$, $\frac{2}{q}$, $\frac{3}{q}$, $\cdots$, $\frac{q-1}{q}$.

(1) Suppose q to be an odd prime. Then the quantities $\frac{n}{q}$ ($n = 1, 2, 3, \cdots, q-1$) are irreducible fractions, and $q - 1$ must be an even number, say $2k$. Then

$$\sum_{n=1}^{q-1}\frac{n}{q} = \frac{1+2+3+\cdots+(q-1)}{q} = \frac{q-1}{2} = k.$$

Hence, in such a sequence, there are $2k$ terms, and the sum of the terms is k.

(2) Suppose that q is an even number of the form 2^rp, where p is any odd prime except one. All the fractions with even numerators are then reducible, and the sequence is

$$\frac{1}{2^rp}, \frac{3}{2^rp}, \frac{5}{2^rp}, \cdots, \frac{2^rp-1}{2^rp}.$$

In this sequence, at least one term is reducible, viz., $\frac{p}{2^rp}$. If there are more, they are the terms $\frac{p}{2^rp}$, $\frac{3p}{2^rp}$, $\frac{5p}{2^rp}$, $\cdots$ up to the greatest odd multiple of p less than 2^rp. This odd multiple must therefore be of the form $4k+1$. Hence an odd number of terms are reducible. Furthermore, the numerator of the last term of the sequence, 2^rp-1, is of the form $4k+1$, so that there is an odd number of terms in the sequence. Rejecting the reducible terms, therefore, leaves an even number of irreducible terms.

If $r = 1$, the number of terms is $p - 1$, an even number. The sum of the terms is then

GEOMETRY.

415. Proposed by W. D. CAIRNS, Oberlin, Ohio.

For three points P_1, P_2 and P_3 on a straight line

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

$$P_2P_3 = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2},$$

$$P_1P_3 = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2},$$

while $P_1P_2 + P_2P_3 = P_1P_3$; prove directly that the sum of the first two radical expressions equals the third.

SOLUTION BY THE PROPOSER.

Introducing a parameter notation, the equation of the straight line $P_1P_2P_3$ may be written

$$\frac{x - x'}{a} = \frac{y - y'}{b} = \frac{z - z'}{c} = \lambda,$$

or $x = x' + \lambda a$, $y = y' + \lambda b$, etc.; whence, for the three given points (as indicated by subscripts),

$$x_1 = x' + \lambda_1 a, \quad y_1 = y' + \lambda_1 b, \quad \text{etc.};$$

$$x_2 = x' + \lambda_2 a, \quad \text{etc.,} \quad \text{etc.};$$

$$x_3 = x' + \lambda_3 a, \quad \text{etc.,} \quad \text{etc.}$$

The substitution of this notation in the given radical expressions gives them the form

$$(\lambda_2 - \lambda_1)\sqrt{a^2 + b^2 + c^2},$$

$$(\lambda_3 - \lambda_2)\sqrt{a^2 + b^2 + c^2},$$

$$(\lambda_3 - \lambda_1)\sqrt{a^2 + b^2 + c^2}.$$

Whence the desired result is evident.

Also solved by H. C. FEEMSTER, R. M. MATHEWS, A. M. HARDING, JAMES A. BULLARD, and C. A. BARNHART.

416. Proposed by W. H. BUSSEY, University of Minnesota.

Let AB be a diameter of any circle, and let C be the third vertex of the equilateral triangle of which AB is one side. Divide AB into six equal parts and let P denote the second point of division from A (*i. e.*, $AP = \frac{1}{3}AB$). Draw the straight line CP and produce it to meet the circle in R. Prove that AR is one-sixth of the circumference.

SOLUTION BY E. L. BROWN, Denver, Colorado.

Let O be the center of circle. Draw OD perpendicular to CP. In right triangle COP, $PO = r/3$, $CO = r\sqrt{3}$, $CP = 2r/3\sqrt{7}$, $PD = r/6\sqrt{7}$, $CD = 9r/2\sqrt{7}$, $OD^2 = 3r^2/28$. In right triangle ODR, $DR = 5r/2\sqrt{7}$.

$$PR = DR - DP = r\sqrt{7}/3 = \tfrac{1}{2}CP.$$

But $PA = \frac{1}{2}PB$. Therefore triangles APR and CPB are similar. Hence $AR = \frac{1}{2}BC = r$.

Also solved by S. LEFSCHETZ, A. L. MCCARTY, GEORGE N. BAUER, HERMON L. SLOBIN, BARNEM LIBBY, RICHARD MORRIS, E. S. INGHAM, and HARVEY MANN.

MISCELLANEOUS QUESTIONS.

Edited by R. D. Carmichael.

That workers in a given field of science or in closely related fields may render an important service by coöperation is one of the commonplaces in the practice of scientific discovery. This coöperation is realized in part through the general meetings in which those interested in the same or related subjects exchange views and present the results of their individual investigations. Those who are thrown together in any way further the same end through conversation; and useful exchange of thought often takes place by correspondence also.

It appears that there is an opportunity to enlarge this coöperation, and thus secure to science a greater advantage from it, by having some medium through which a certain type of questions can be presented to one's co-laborers and their reactions promptly secured. In the field of the mathematical sciences, one thinks immediately of the problems in *L'Intermédiaire des Mathématiciens*. We propose to establish in this journal also a department in which mathematical questions of this general kind are treated. From the account of our purposes, which we give below, it will be seen that we shall supply a need different from that which is provided in the *Intermédiaire*. In fact, so far as we know, there does not at present exist a medium of coöperation of the kind which we are about to establish.

Through this department of "Miscellaneous Questions" two purposes are to be served. One of these pertains more or less to pedagogical matters; the other relates entirely to scientific questions of a mathematical nature. As to our purposes in the former the reader is referred to the concluding paragraph under the heading "Problems and Questions" in the October number of the Monthly, from which the following is quoted:

> "Such questions might well refer, for example, to the selection of subject matter and method of presentation in courses in collegiate mathematics, and they might elicit replies in the form of a short symposium or they might even lead in some cases to extended articles. On the other hand, these questions might refer to difficulties of interpretation or to obscure steps in some demonstration or to anything whatsoever of mathematical import on which the questioner honestly seeks light."

As to the latter, the following remarks will explain our point of view: In many investigations both in mathematics and in physics there arise from time to time certain definite mathematical problems which are incidental to the main research in hand and must be solved before that research can be carried forward to completion. These problems often give special difficulty. Sometimes the investigator would be glad to have the assistance of others in their solution, either because they are difficult or because he prefers not to divert his attention from the main problem to solve the incidental one which arises in this way. One principal aim of this department is to secure and publish just such problems and their solutions. We shall also welcome similar ones which sometimes arise in connection with one's reading.

The mutual service which students and investigators in mathematics may render each other through such a medium is apparent to all who know the peculiar difficulties which such persons have to face. There is also another end of equal (and perhaps greater) importance which we hope may be served by this department. We desire to have proposed here well-defined mathematical problems which arise in physical investigations. To call wide attention to such problems will serve two useful purposes.

In the first place, the physicist may sometimes draw on the mathematician for assistance from his fund of knowledge about pure mathematics, and thus often secure readily the solution of his special incidental problem or an indication of its essential nature and difficulty. Thus, in the end, there may well be a considerable saving of time and energy.

In the second place, the mathematician cannot fail to profit largely by this interchange, as any one must see who remembers how often some important mathematical theory has been brought to notice and development through the persistent demand of physics for the means of solving her problems.

When a problem has been proposed to this department we shall undertake to supply the proposer with a solution as soon as possible after one has been received by us—sending it to him in advance of publication when that is feasible and is desired.

The undersigned will attempt to ascertain whether the problems which come to us are of sufficient interest or difficulty to justify examination by several persons. If some ready means of solving a particular problem is found, the proposer will be so informed as promptly as possible and a solution will usually be furnished him privately; and in such event the problem will be published only when it seems to be of interest in itself.

The first requisite for the usefulness of this department is to have problems and questions proposed; and, consequently, persons interested are urgently desired to send in their proposals at their early convenience. These should be accompanied by any remarks which may be useful to the editor in stating the question so that the reader may see why it is of interest to have an answer. Every communication should be signed by the author; his name, however, will be withheld if he so requests.

While the plans for this department were being considered two communications came to us, which are of the general nature wanted here; consequently, the questions involved are printed in this issue as 1 and 2. Answers are desired both to these questions and to others to be published later. It is obvious that a large part of the usefulness of this department will depend on the promptness with which answers to questions are supplied; and consequently our readers are requested to communicate replies as early as possible after a question is published.

All correspondence relating to this department should be addressed to R. D. Carmichael, 417 N. Lincoln Street, Bloomington, Indiana.

QUESTIONS.

1. In connection with the investigation of a problem in physics MR. LOUIS COHEN, Washington, D. C., requires to have expressed in the form of a Fourier's series,

(1) $$y = \Sigma A_n \cos nt + \Sigma B_n \sin nt,$$

the general solution of the differential equation

(2) $$\frac{dy}{dt} + (\alpha + \beta \cos t)y = \rho \cos t,$$

where α, β and ρ are constants. Is the solution of this problem contained directly in matter which has already been published? If not, how may one proceed to the determination of the constants A_n and B_n so that (1) shall be the general solution of (2)?

2. It is clear, on the most casual observation, that the average college curriculum in mathematics is unbalanced in two respects: (1) algebra predominates over geometry, (2) analytic geometry predominates over our synthetic geometry. This reacts in two undesirable ways: (1) to deprive the college student of a rich and interesting field of study, (2) to give a very one-sided training for prospective teachers of high school mathematics. What can be done to remedy this situation?

Note.—One very interesting and thought-arousing answer is given to this question in the paper by Professor Bussey in this issue. We shall welcome discussion along this line.—MANAGING EDITOR.

NOTES AND NEWS.

FLORIAN CAJORI, CHAIRMAN OF THE COMMITTEE.

Attention is called to questions one and two in this issue under the new department "Miscellaneous Questions," and to the introduction to this department by Professor R. D. Carmichael, who will have it in charge.

Dr. S. E. Rasor, of the Ohio State University, has been promoted to a full professorship in mathematics.

The Deutsche Mathematiker-Vereinigung held its annual meeting in Vienna on September 22–25, 1913. Altogether thirty-seven papers were presented.

Mr. George I. Gavett, University of Washington, has accepted an assistantship in the University of California for the present year.

At Lehigh University, Joseph B. Reynolds has been made assistant professor of mathematics and astronomy.

Mr. C. W. Wester, graduate student at the University of Chicago, has been appointed to an instructorship in mathematics at the University of Iowa.

At the University of Colorado the following have been appointed instructors in mathematics for the coming year: Florence Kendall, B.A., University of Colorado, 1912; W. E. Edington, B.A., Indiana State Normal, and last year a scholarship student at the University of Chicago, majoring in mathematics.

Dr. H. L. Rietz, associate professor of mathematics in the University of Illinois, has been promoted to a full professorship in the same institution. Mr. H. C. Zeis has been appointed assistant in mathematics in place of Mr. G. E. Carscallen, who accepted a professorship in Hiram College.

The Proceedings of the Fifth International Congress of Mathematicians held at Cambridge in August, 1912, have been edited by the general secretaries of the Congress, E. W. Hobson, professor of pure mathematics in the University of Cambridge, and A. E. H. Love, professor of natural philosophy in the University of Oxford.

The Proceedings appear in two volumes; price 30 shillings.

On the 25th of last May, the "friends and admirers, collaborators and students" of Professor Felix Klein presented to him his protrait, painted by Max Liebermann. The published presentation address is followed by a large number of names of mathematicians from various countries. The names of the following Americans appear in the list: M. Bôcher, F. N. Cole, P. Field, J. C. Fields, F. Franklin, A. B. Frizell, M. W. Haskell, G. A. Miller, E. H. Moore, W. F. Osgood, D. E. Smith, V. Snyder, J. H. Tanner, H. W. Tyler, E. B. VanVleck and F. S. Woods. In reply to this address Professor Klein stated that the real motive of his life had always been the collaboration with those engaged in similar work.

Professor J. F. Downey, University of Minnesota, expects to retire at the end of the present academic year. He is professor of mathematics and dean of the college of science, literature, and arts.

The Spanish Mathematical Society, which was organized at Madrid a little more than three years ago, has now 470 members, according to the July, 1913, number of *Revista de la Sociedad Matematica Española*. The membership fee for foreign members is 18 pesetas (about $3.60) per year. Members receive without extra charge a copy of the "Revista." This elementary journal appears monthly in the Spanish language, and is published for the society by an editorial committee and the secretaries. The headquarters of the society are at San Bernardo, 51, Madrid, Spain.

The 26th fascicule of the *Encyclopédie des Sciences Mathématiques* was published on June 30, 1913. It contains 160 pages, and is devoted to partial differential equations. The first 55 pages, edited by G. Floquet of Nancy, are concerned with the study of general properties and linear equations of the first order. The remainder is edited by E. Goursat of Paris, and is devoted to non-linear equations of the first order, and to equations of a higher order.

We learn from the widow of Dr. Alexander Macfarlane, of Chatham, Ontario, that he died on August 28, 1913, after an illness of six months from valvular heart disease. He was at one time instructor in physics in the University of Edinburgh and later held the professorship of physics in the University of Texas. He had been a subscriber to the MONTHLY from the beginning, and one of its warmest supporters. Our sympathies are extended to Mrs. Macfarlane in her bereavement.

At Rutgers College, Professor Richard Morris has been given the title "professor of mathematics" in place of "professor of mathematics and graphics." Also two new appointments have been made to the rank of assistant professor, namely, Dr. W. B. Stone and Dr. S. E. Brasefield.

Announcement has been made by the department of mathematics, Adelbert College, Western Reserve University, of a course in the history and teaching of elementary mathematics for 1913–14. The course is intended especially for instructors in elementary mathematics and persons preparing to become teachers, and is open to both men and women.

The death is announced of Mr. T. N. HAAN, of Mohawk, Tenn. He was an enthusiastic supporter of the MONTHLY and was interested in mathematics especially on the practical side. He left an extensive library and a manuscript on mathematical lines which he had prepared for publication.

The mathematics section of the California High School Teacher's Association held two meetings during the Conference in Berkeley last July. At the first meeting a committee was appointed to devise means of raising the standard of scholarship throughout the state. This committee recommended that, in the larger cities, circles be formed for the study and solution of problems. Also that every teacher become familiar with at least one first class journal, such as the AMERICAN MATHEMATICAL MONTHLY, and if possible take up the study of *Mathematical Monographs*, edited by J. W. A. Young. As a result of this recommendation, the study of two or more of these monographs is included among the subjects in which examinations will be given in the university extension department of the University of California. The second meeting emphasized the need for arithmetic which concerns the investments and expenditures of daily life. It constituted a plea for more such courses in the high school to meet the future needs of the average boy and girl. Dr. HENRY W. STAGER, of the Junior College at Fresno, was elected president for the ensuing year.

As the MONTHLY is nearing the end of the first year in its new form, attention is called to the classification of articles so far published to be found on the supplementary pages of the last issue. Readers will be pleased to know that the subscription list has been nearly doubled during the year. However, this is only a beginning and the editors bespeak the coöperation of all friends in still further extending the list. An important field where such influence may be effective is in connection with school and college libraries and public libraries. Sample copies of this issue have been sent to a large number of such libraries not now on the mailing list.

It will greatly facilitate the business management if all renewals for 1914 may be received by the Treasurer before January. Please note that the date "Jan. 14," on the address label signifies the expiration of your subscription with the December issue.

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VOLUME XX DECEMBER, 1913 NUMBER 10

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THE AMERICAN MATHEMATICAL MONTHLY

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NUMBER SYSTEMS OF THE NORTH AMERICAN INDIANS.

BY W. C. EELS, Whitworth College.

II. SYSTEMS OF NUMERATION.

Counting cannot be carried far by the use of successive unrelated terms or symbols for each number. For higher terms some system of compounding is necessary, the usual method being by reference to some stopping point or *base* which is thought of as a new starting point. The choice of this base is of fundamental importance in the development of a true *number system* as distinguished from a mere *series of numbers*. The choice of this base seems usually to be related to primitive finger counting. One hand, two hands, or all the fingers and toes, are the three most natural stopping points, leading respectively to quinary, decimal, and vigesimal systems as illustrated below. There is much variation in the choice of these bases and in the way in which they are built up into systems for actual use after the base has been established. Thus a pure quinary system is rare, usually merging into a decimal or vigesimal one for the higher numbers. Accordingly in the list given below of the number of instances of each system found, quinary and quinary-decimal systems have been classed together, and similarly vigesimal and quinary-vigesimal.[1] The numbers found are: decimal 146, quinary and quinary-decimal 106, vigesimal and quinary-vigesimal 35, quaternary 15, ternary 3, octonary 1 (binary 81). The binary instances given in parantheses refer simply to languages in which the duplicative principle (already discussed) occurs. These may be thought of as traces of a binary system. But there are no binary systems in the true sense of the word.

1. Decimal Systems. Many of the American Indian systems have decimal scales as regular and as complete as our own, extending to quite high limits.

[1] Probably several systems have been wrongly classified, due to inadequate data. Many authorities give the numerals from 1 to 10 and then for 100, or perhaps by tens to 100. The greatest error is probably in the classification of too many languages as decimal. This has been done in some cases from comparison with cognate languages, where the direct data were insufficient. Some have been called quinary-decimal, where further data would probably show they were actually quinary-vigesimal. These variations in detail will probably not seriously affect the general conclusions stated.

Many others, while predominatingly decimal, are combined with other systems, decimal-quinary and decimal-vigesimal being most frequent. The decimal system appears exclusively (i. e., no elements of other systems have been found) in only eleven families, all of which are small ones totaling only 19 languages of those examined. Thus in only a comparatively few cases do the languages of a single family use the decimal system consistently.

As illustrations we find "one hundred" expressed by a unique word or by such forms as "completed," "stock of tens," "10×10"; "thousand" by "$10 \times 10 \times 10$," "10×100," "big hundred," "old man hundred," "large stock of tens"; "million" by "$1{,}000 \times 1{,}000$," "big thousand," "too many to count." While there are many variations, intermediate numbers as a rule are formed as in English.

2. **Quinary and Quinary-decimal Systems.** As already mentioned a pure quinary system is very rare, if indeed one exists. The pure form would require only five elements and is of course 1, 2, 3, 4, 5, $5 + 1$, $5 + 2$, $5 + 3$, $5 + 4$, 2×5, and so on to 3×5, 4×5 and 5×5 as a new primary base. There are various approximations to such a system. Even 10 as "2×5" is uncommon although several instances of this have already been given. We have as variations for the numbers from 6 to 9, $6 = X + 1$ ("X" standing for some descriptive, non-numerical word), $7 = X + 2$, etc., arising from such forms of thought as "again 1," "second 1," "1 more," "on the other, 1," the numerals of the second quintate repeating without the use of the expressed base *five*. One or more of these forms may be entirely lacking, 8 being expressed "2×4" and similarly. But only in case at least two of the numbers between 5 and 10 show such a formation have we classed it as quinary. *Nine* is a frequent variation, since it is so often expressed by the subtractive principle. With these explanations we may state that quinary systems occur in about one third of the languages examined, appearing most frequently in the region around the Gulf of Mexico and least in the languages along the Northwest Coast.

Space will not permit illustrative systems to be given in full. Luiseno is purely quinary to ten,[1] Gallinomero has only a few variant forms until it reaches forty, where it changes to decimal,[2] while most of the quinary systems are of the general type, 1, 2, 3, 4, 5, $X + 1$, $X + 2$, $X + 3$, $X + 4$ and a new word for ten, e. g. Delaware[3] or Shawnee.[4]

3. **Vigesimal and Quinary-vigesimal Systems.** Vigesimal systems more or less complete occur in about one tenth of the languages examined. With the exception of the Caddoan family in the middle west they all appear along the Pacific Coast or in the far north. The obvious explanation of this system is in the digital origin of counting. The well known ethnologist Gatschet says that

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[1] Dixon & Kroeber: "Numeral systems of languages of California," in *Amer. Anthropologist*, N.S., Vol. 9, p. 681.

[2] Dixon & Kroeber: *op. cit.*, p. 676.

[3] Zeisberger, D., "Grammar of the Delaware," in *Amer. Philosophical Soc. Trans.*, N.S., Vol. 3, p. 65.

[4] Johnson, J., "Present State of Indian Tribes Inhabiting Ohio," in *Amer. Antiquarian Soc. Proc.*, Vol. 1, p. 269.

"tribes living in tropical and hot climates mostly possess the vigesimal system of notation, which is rather infrequent among the Indians of the United States."[1] He finds the explanation in the fact that they live barefoot as contrasted with the moccasined northern Indians. But this "barefoot" explanation, also given by other writers (especially to account for the unusually well developed vigesimal systems of the Mexicans and Aztecs),[2] rather breaks down in the case of the Eskimo and tribes of the north Pacific Coast where the climate is scarcely adapted to the barefoot stage. And yet among the Eskimo the vigesimal system has found its fullest development north of Mexico. The use of fingers and toes for the development of a vigesimal system seems to be independent of climate.

Twenty is the primary basis of the vigesimal system and is usually expressed as "man," "Indian," "man completed"; the multiples of twenty being expressed as "two men," "three Indians," etc. Pawnee carries this to 1,000 which is "50 persons."[3] The vigesimal system usually occurs in connection with some other system. Three types of combination may be noted.

(*a*) *Quinary-vigesimal.* This is most frequent. The Greenland Eskimo says "other hand two" for 7, "first foot two" for 12, "other foot two" for 17 and similar combinations to 20, "man ended." The Unalit is also quinary to twenty which is "man completed." But 40 is "two sets of animals' paws," 60 "three sets of animals' paws" and so on regularly to 400 where there is an interesting change in the formation of this primary base (20×20) from animals back to man, for 400 is "20 sets of *man's* paws."[4]

(*b*) *Decimal-vigesimal.* Systems in which no quinary elements are found are comparatively rare. Wintun is alternately decimal and vigesimal, 20, 40, 60, being "one Indian," "two Indian," "three Indian," while 30, 50 are "3×10," "5×10."[5] Others are similar.

(*c*) *Quinary-decimal-vigesimal.* Several systems show a combination of the three digital bases in their formation. Kopiagmiut is quinary to ten, decimal for the formation of the odd tens and vigesimal for the even ones.[6] Amador is purely decimal to ten, quinary from ten to twenty, and then vigesimal, the odd tens being formed by addition of ten to the preceding even ones, e. g., $50 = 40 + 10$.[7] Haida is quinary-decimal and quinary-vigesimal alternately to a hundred, then pure vigesimal, 400 being $20 \times 20 \times 1$ and 800 being $20 \times 20 \times 2$.[8]

4. **Quaternary Systems.** Fairly well defined quaternary systems reaching

[1] Gatschet, A. S.: "Indian Numeral Adjectives," in *Amer. Antiquarian*, Vol. 2, p. 210.

[2] The best developed vigesimal system on the American continent is probably the Maya of Mexico. This is given and discussed in Conant's *Number Concept* and in much fuller form by Thomas in the 19th Rpt. Bur. of Eth., p. 853.

[3] Dunbar, J. B.: *The Pawnee Language*, 1893. (No place of publication given. Pamphlet, Copy in Newberry Library, Chicago.)

[4] Nelson, E. W.: "Eskimo about Bering Strait," in 18th Ann. Rpt. Bur. of Eth., 1896–7, p. 238.

[5] Dixon & Kroeber: *op. cit.*, p. 675.

[6] Petitot, P.: *Vocabulaire Francaise Esquimau*, Paris, 1876.

[7] Dixon & Kroeber: *op. cit.*, p. 680.

[8] Harrison, C.: "Haida Grammar," in *Royal Soc. of Canada Proc. and Trans.*, Vol. 1, 2d Ser., Sect. 2, p. 123.

to eight may be found among the Montagnais of the far north, the Foxes of Wisconsin, the Iowas and Missouris of the Plains—but they find their best and fullest development into true systems in various California tribes. Usually they are mixed with other systems, but one or two cases are found of practically pure quaternary systems.

It is not easy to account for the origin of such a system. Two possibilities may be suggested. (*a*) *Digital.* Perhaps a few tribes, for reasons best known to themselves, did not use their thumbs in counting. This is possible but there is little if any linguistic or observational evidence to support it. Most frequently these systems show the words "stick," "middle," "body." "Body" might be considered digital. (*b*) *Sacred Number Theory.* Four was the sacred number of many widely separated tribes of Indians. The four cardinal points of the compass and the four seasons were recognized by the Indians. Among several tribes the literal meaning of "four" is "complete," "right," "perfect." However the systems of these tribes are not quaternary ones. A chart for the sacred rites of the Ojibwas shows four degrees of initiation for medicine men. The swastika, with its four arms, originated with the Indians of the Southwest and was much used in basketry. There is a widespread "four worlds" of Indian mythology. In the various languages spoken on Puget Sound the same word for four, with minor variations, is seen most frequently, and it is the only number word common to about a dozen of these languages which have been most carefully studied by a missionary among them. All these data are suggestive of the origin of a quaternary system, although the fact remains that most of the instances given above are from tribes which do not actually possess such systems.

Santa Barbara as far as sixteen is as follows: 1, 2, 3, 4, $X + 1$, $X + 2$, $X + 3$, 8, 9, $X + 2$, 11, 3×4, $X + 1$, $X + 2$, $X + 3$, 16. For 20 and above it is decimal, probably due to contact with civilization. San Luis Obisbo has $X + 2$, $X + 3$, for 6, 7; $8 + 1$ for 9; $12 + 1$, $12 + 2$, $12 + 3$ for 13, 14, 15.[1] Numerous other languages have many quaternary forms.

5. **Ternary System.** A ternary system is much rarer than a quaternary and nowhere occurs in a pure form. Two well-developed ones are given below. It is even more difficult than in the case of the quaternary system to account for its origin. It is possible that to some primitive minds it seemed natural to count one, two, three, and then by groups of threes. In some languages there is a similarity between the words for "this," "that," "that (remote)" or "here," "there," "yonder" and the first three numerals. A number of instances of 6 expressed as 2×3 have already been given[2] and some writers have mentioned them as examples of a ternary system. But for reasons already explained they are better considered as formed by the Duplicative principle, unless occurring in connection with nine or twelve similarly formed. The Cuchan numerals for 3, 6, 9, are ha-mook, hum-hook, hum-ha-mook.[3] In San Antonio 4 is related to 1, 6 is derived from 3, and 12 is directly 4×3, 15 is 5×3, and 13 is $12 + 1$.[4]

[1] Dixon & Kroeber: *op. cit.*, p. 682.
[2] See "Duplicative Principle" already discussed.
[3] Trumbull: *op. cit.*, p. 41.
[4] Dixon & Kroeber: *op. cit.*, pp. 683, 690.

But Coahuiltecan of Texas is the most interesting example found. It has binary, ternary, quaternary, quinary, decimal and vigesimal features,[1] but seems to be prevailingly ternary. Its system as far as 50 is as follows:

1	4	$7 = 4 + 3$	$10 = 5 \times 2$
2	5	$8 = 4 \times 2$	$11 = 10 + 1$
$3 = 2 + 1$	$6 = 3 \times 2$	$9 = 4 + 5$	$12 = 4 \times 3$
$13 = 12 + 1$	$16 = 15 + 1$	$19 = 18 + 1$	$30 = 20 + 10$
$14 = 12 + 2$	$17 = 15 + 2$	20	$40 = 20 \times 2$
$15 = 5 \times 3$	$18 = 6 \times 3$?		$50 = 40 + 10$

6. **Octonary System.** A single system with a base eight is known, the Yuki of California. This interesting system in translation is:[2]

1	9 = beyond-1-hang	17 = 1-middle-project
2	10 = beyond-2-body	18 = 2-middle-project
3	11 = -3-body	19 = 3-middle-project
4	12 = -4-body	20 = 4-middle-project
5	13 = -5-body	
6	14 = -6-body	
7	15 = -7-body	
8	16 = middle-none	

7. **Traces of Other Systems.** Only traces of the use of other bases are found. (*a*) *Binary.* If we include the large number of examples of the duplicative principle already discussed, we have many instances of binary elements. But these are only in the multiplicative principle and refer simply to doubling. The only place where we have found the additive principle used is in the Coahuiltecan (just given) where $3 = 2 + 1$. We may also notice Chutsinni: 2, *stunga;* 4, *stung-sung;* 8, *stun-sunga.* Yokaia: 2, *ko;* 4, *duo-ko;* 8, *ko-ko-dol.* (*b*) *Sexanary.* Aside from the instances of twelve expressed as 2×6 mentioned under the duplicative principle, we have Wimunche Ute: $7 = 6 + 1$, Rumsen: $7 = X + 6$, $8 = 2 + 6$. (*c*) *Base of Nine.* Trinity: $10 = 9 + 1$, $11 = 9 + 2$. (*d*) *Base of Forty.* Chwachamaju: 40, *ku-hai,* "1-stick"; 80, *ko-hai,* "2-stick." (*e*) *Base of Sixty.* Achomawi: $70 = 60 + 10$, $80 = 60 + 20$.

8. **Conclusions.** The most striking feature of the systems which have been studied is their diversity, even in languages of the same family, and much more marked when the country as a whole is considered. For instance in the closely related languages of the Yukian family in California, although the numerals from one to four are quite similar, yet two of the systems are quinary-decimal, a third is quinary-vigesimal, while the fourth is octonary; or in the Pujunan family in which one system is decimal, eight quinary-decimal and two quinary-vigesimal.

How many unique abstract number words are necessary for building up a number system? We recall that in English ten are used and that all up to one hundred are but combinations of these ten. Of course two are sufficient—in a binary system. Many of the Eskimo tribes manage quite well with five. The Luiseno of California has but five abstract number words, but it has higher units

[1] Gallatin, A.: *Trans. Amer. Ethnological Soc.*, Vol. 1, N. Y., 1845, p. 1.

[2] Dixon & Kroeber: *op. cit.*, pp. 677, 685. This system is given by these authors as quaternary, but they have acknowledged that this is an error, and that it should be classed as octonary.

which are chiefly descriptive phrases indicating various combinations of hands and feet. Many languages which have a decimal system get along easily without the full quota of ten as used in English. This is accomplished by the use of such combinations as $8 = 2 \times 4$, $6 = 2 \times 3$, 7, 8, 9 as 3, 2, 1 respectively subtracted from 10, and others which have already been given. The Coahuiltecan (given under the ternary system) forms all the numbers up to twenty with only four unique words, those for 1, 2, 4, 5—a very remarkable instance.

III. Miscellaneous Points.

1. **Limits in Use.** The numerals in some languages are given as high as a million and in many others to a thousand or more. We are naturally led to inquire whether such high numbers were actually in use by primitive people. The evidence found on this point cannot be here given in detail. Only a few conclusions may be stated. The Eskimo seem to be poorest in ability to count, being low in comparison with the tribes of the United States. Most of them in ordinary conversation do not use above five or six, referring to higher numbers as "many," but the more intelligent of them can count to 400. The Indians of the eastern United States who stand comparatively high intellectually, could use their numerals to 10,000 and their systems were such as to admit of indefinite expansion, one billion for instance being expressed as $1{,}000 \times 1{,}000 \times 1{,}000$. The Crees could count correctly as far as 1,000. The Winnebagoes are said to use their numerals as high as one million Indefinite and countless numbers they represent by the terms "leaves on the trees," "stars of the heavens," "blades of grass on the prairie," "sand on the lake shore." The Crows do not count above a thousand, as they say honest people have no use for higher numerals! The Apaches cannot use numbers beyond 100,000. Any of the California tribes of which positive statements can be made can count into the hundreds. As a definite upper limit to counting ability we find that the Tuolomne are credited with the ability "to count with great rapidity almost to infinity"![1]

Speaking in general terms, we may say that the rather highly developed Siouan tribes of the Plains, the Iroquoian and Algonquian families of the east, and the Muskogean tribes of the South—all rather high in the scale of civilization—could count intelligently at least into hundreds of thousands and had words for even higher numbers; that most of the other Indians of the country had little actual knowledge of forms for numbers higher than thousands; while the Eskimo of the far north were limited to hundreds and in many cases to twenty or even ten. It is probable that before coming in contact with European civilization the Indians had little occasion to use numbers beyond a thousand. But the systems of many of them were such as to admit of indefinite and easy extension when needed.

2. **Numeral Classifiers.** Some tribes use different sets of numerals for counting different classes of objects. The Tsimsian language has quite distinct forms for counting *men* and for counting *things*. More frequently the number stem is

[1] Johnson, A.: In Schoolcraft's *Indian Tribes* (*op. cit.*), Vol. IV, p. 406.

modified by a prefix or suffix which is in the nature of a classifier to denote the class of objects counted. The Haida has no less than 15 of these classifiers. Others seem to have an even larger number.[1] This usage is found very generally among the languages of the north Pacific coast and but rarely in other parts of the country. The Tsimsian mentioned above has different forms for abstract counting, for counting flat objects or animals, for round objects or time, for men, for long objects, for canoes, for measures.

3. **Verbal Nature.** In a few languages the numerals are true verbs instead of adjectives, and as such are conjugated through all the variations of mood, tense, person and number. As far as found this peculiarity is limited to three languages, Cree, Crow, Micmac.

4. **Derivative Numerals.** The formation of ordinals, adverbials and distributives from the cardinal numerals is more properly a grammatical than a numerical process and as such need receive only slight notice here. (*a*) *Ordinals* are found most frequently, usually being formed from the cardinals by a suffix or other terminal modification, occasionally by a prefix. In the Creek an unusual method is found, the ordinal being formed from the cardinal in the same way that the superlative of the adjective is formed from the comparative. (*b*) *Distributives* are also frequently found. They are formed from the cardinals by prefixes, suffixes, or reduplication. (*c*) *Adverbials,* as far as noted, are formed by suffixes.

5. **Arithmetical Operations.** We shall close our discussion of the varied, interesting, and intricate number systems of the North American Indians with a reference to their ability to perform arithmetical operations. In our study of the principles of formation of number words we found an extensive use of addition and multiplication, a less use of subtraction, and very slight use of division in the formation of number systems. But aside from these instances (all operations on only the *bases* of the systems) the calculative ability of the American Indians was very slight and of the most elementary sort. Addition, subtraction or multiplication was accomplished only with the aid of the fingers, sticks, pebbles or other convenient counters. It is probable that the native Indian mind had practically no idea of mental arithmetic, being unable to multiply or divide numbers mentally, or even to add or subtract any except the smallest. His need for such operations was probably as slight as his knowledge.

THE CURVE OF LIGHT ON A CORRUGATED DOME.

By WM. H. ROEVER, Washington University.

The Phenomenon. On a bright day a person situated at a high point, within a distance of three or four miles of the new Roman Catholic Cathedral of St. Louis, will observe a well-defined curve of light on the dome of the cathedral. This curve, which is closed, passes through the highest point of the dome and

[1] The Maya of Mexico has at least 75 such classifiers. See article by Mrs. Z. Nuttall in *Amer. Anthropologist,* N.S., Vol. 5, p. 667.

changes its size and position with the changing position of the sun. The accompanying photograph (Fig. 1), taken from a high building about a mile from the Cathedral, shows this closed curve which resembles a halo. The same sort of phenomenon may be observed on the stem of a watch when held in the light.

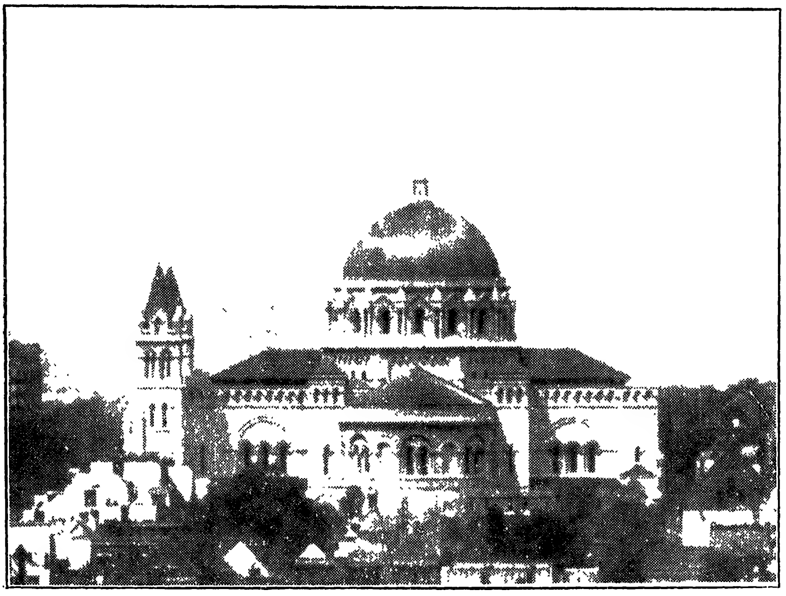


FIG. 1.

The Cause. The dome of the cathedral is covered with glazed tiles laid along those great circles of the spherical dome which pass through the highest point, thus forming a sort of corrugated surface, of which the ridges lie

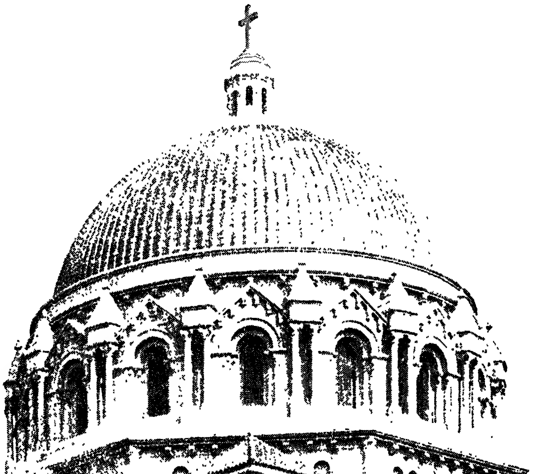


FIG. 2.

along the meridians of the sphere (Fig. 2). The sun's light is reflected from this glazed, corrugated surface, and an observer, properly situated, sees in each corrugation a reflected image of the sun or an actual brilliant point, as this image is sometimes called. The assemblage of these images or brilliant points, any two consecutive ones of which are too near each other to appear separated to the eye of the distant observer, forms the curve or halo described above.

The Mathematical Problem Involved. In order to determine the mathematical nature of this curve of light we must make some assumptions, as we do in the mathematical formulation of every physical problem. It is assumed that the dome is a portion of a spherical surface, in fact, a hemisphere, and that the meridional corrugations are so small and numerous that we may regard the spherical dome as covered with a family of polished wires which are laid along the great circles through the highest point. It is further assumed that these wires are so fine that they may be regarded as mathematical curves. Furthermore, the sun's rays are regarded as a system of parallel rays, and the observer is supposed to be far enough away from the Cathedral so that the reflected rays which reach his eye may also be regarded as belonging to a system of parallel rays. Under these assumptions the mathematical problem involved may be stated as follows:

Determine the locus of the actual brilliant points of the meridians of a sphere with respect to a source of light which is infinitely distant and an observer who is (practically) infinitely distant.

Solution of the Mathematical Problem.

Definition. A point P is said to be an *actual brilliant point* of the curve C (plane or twisted) with respect to the source of light P_1 and an observer P_2 when

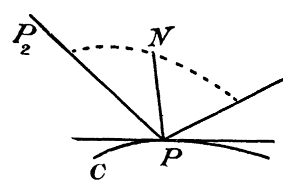


Fig. 3.

the internal bisector of the plane angle P_1PP_2 is a normal to the curve C at the point P. In particular P_1 and P_2 may be infinitely distant.

Accordingly, if we denote by l_1, m_1, n_1 the direction cosines of the directed line PP_1, by l_2, m_2, n_2 the direction cosines of the directed line PP_2, and by l, m, n numbers proportional to the direction cosines of the tangent line to the curve C at the point P, then the condition that the point P should be an actual brilliant point of the curve C, with respect to the source P_1, and the observer P_2, is*

$$(1)\qquad (l_1 + l_2)l + (m_1 + m_2)m + (n_1 + n_2)n = 0.$$

* For more details see the author's paper "Bril ant Points of Curves and Surfaces," *Transactions of the American Mathematical Society*, Vol. 9, No. 2, pp. 245–279.

In our particular problem the source P_1 is the infinitely distant sun, P_2 is the eye of the (practically) infinitely distant observer and the curve C is a vertical great circle, that is a meridian, of the spherical dome. In order to get the form of condition (1) for our problem, let us assume a set of rectangular axes of which the origin lies at the center of the spherical dome and of which the axis of z is vertical and positive in the direction of the zenith. The equations of a meridian then are (Fig. 4)

$$F = 0 \quad \text{and} \quad \Phi = 0, \tag{2}$$

where

$$F = x^2 + y^2 + z^2 - R^2 \quad \text{and} \quad \Phi = \frac{y}{x} - k,$$

in which R denotes the radius of the spherical dome and k is a parameter whose value depends on the particular meridian under consideration.

The direction cosines of a tangent to the curve (2) at a point (x, y, z) of this curve are proportional to the three determinants of the matrix

$$\left\| \begin{matrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial z} \\[2ex] \dfrac{\partial \Phi}{\partial x} & \dfrac{\partial \Phi}{\partial y} & \dfrac{\partial \Phi}{\partial z} \end{matrix} \right\|.$$

Hence if, as above, we denote by l, m, n numbers proportional to these direction cosines, we have

$$l : m : n = -xz : -yz : x^2 + y^2. \tag{3}$$

Since, for our problem, l_1, l_2, m_1, m_2, n_1, n_2 are constants at any particular instant of time, let us introduce new constants by the relations:[1]

$$\alpha = l_1 + l_2, \quad \beta = m_1 + m_2, \quad \gamma = n_1 + n_2.$$

Hence condition (1) for our problem takes the form:

$$-\alpha xz - \beta yz + \gamma(x^2 + y^2) = 0. \tag{4}$$

This equation represents a quadric cone whose vertex is at the origin of coördinates, and whose horizontal sections are circles. This cone contains as two of its elements the axis of z and the internal bisector of the angle formed at the origin by the rays which pass to the sun and to the observer, and the plane of these two elements is a plane of symmetry. It is the locus of all of the actual brilliant points of the two parameter family of curves obtained by allowing both R and k in equations (2) to be variable parameters. The locus which we are seeking is the intersection of this locus with the sphere:

$$x^2 + y^2 + z^2 - R^2 = 0. \tag{5}$$

Since the dome is the upper hemisphere of the sphere (5), we are interested only

[1] These constants are proportional to the direction cosines of the internal bisector PN of the angle P_1PP_2.

in the upper branch of the curve made by the intersection of (4) and (5). This is the only portion of the locus which we observe. Figure 4 is an orthographic projection in which the sun is represented as being in the xz plane at an altitude of 60° and the observer at an infinite distance in a direction perpendicular to the plane of the drawing.

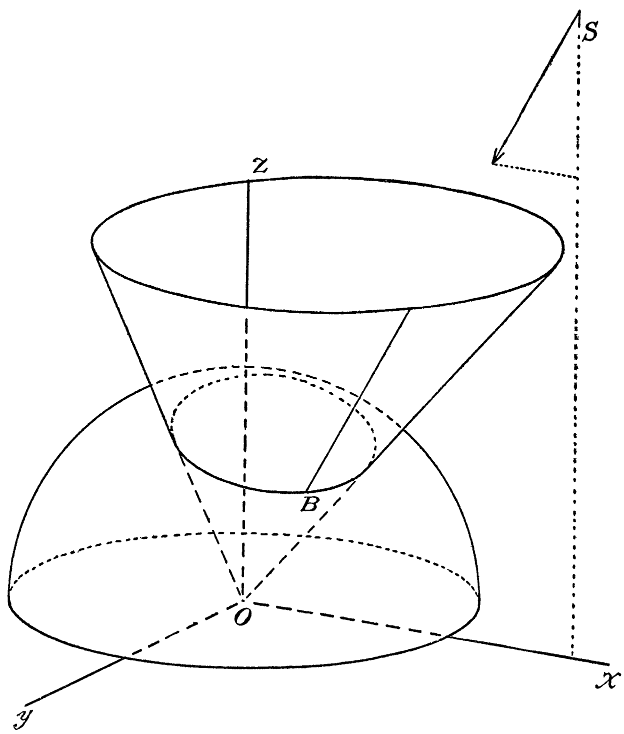


Fig. 4.

The observer is practically in the horizontal plane which passes through the center of the dome. Hence, practically, $n_2 = 0$. When the sun is in the horizon $n_1 = 0$ also, and therefore $\gamma = 0$. Then equation (4) becomes

$$z(\alpha x + \beta y) = 0,$$

and the curve of light (4)(5) degenerates into two great circles of the sphere, of which one is horizontal and the other vertical.

Finally, it is interesting to note that if the corrugations had been parallels instead of meridians, the curve of light would have been a meridian.[1]

Figure 1 shows, in addition to the curve of light on the dome, a line of light on the conical roof of the eastern apse of the Cathedral. This is the element along which a plane perpendicular to the internal bisector of the angle P_1PP_2 (Fig. 3) is tangent to the conical roof.

[1] This will be further elucidated in paper on "Optical Interpretations in Higher Geodesy" soon to be published in the MONTHLY.

ON CERTAIN DIOPHANTINE EQUATIONS HAVING MULTIPLE PARAMETER SOLUTIONS.[1]

By R. D. CARMICHAEL, Indiana University.

1. The object of this note is to obtain a multiple parameter solution of each of several Diophantine equations, as follows:

$$
\begin{array}{rr}
x^3 + y^3 + z^3 = 2t^3, & x^3 + y^3 + z^3 = 2t^{9k}, \\
x^3 + y^3 + z^3 + u^3 = 3t^3, & x^3 + y^3 + z^3 + u^3 = kt^{\alpha}, \\
x^3 + 2y^3 + 3z^3 = t^3, & x^3 + 2y^{3\alpha} + 3z^{3\beta} = t^3, \\
x^4 + y^4 + 4z^4 = t^4, & x^4 + 2y^4 + 2z^4 = t^4, \\
x^4 + 8y^4 + 8z^4 = t^4, & x^4 + y^4 + z^4 = 2t^4,
\end{array}
$$

where k is a positive integer and α and β are odd positive integers. The methods employed are altogether elementary in their character.

2. If in the equation[2]

$$x^3 + y^3 + z^3 = 2t^3, \tag{1}$$

the numbers x, y, z, t have the common divisor d, then the equation may be divided through by d^3 and the resulting equation will be of the same form as (1). Hence it is sufficient to seek only those solutions of (1) in which x, y, z, t have the greatest common divisor unity; and consequently we confine attention to this case. It is obvious, then, that two of the numbers x, y, z, are odd while the third one is even. If we suppose x and y to be odd, we may write

$$x = r + s, \quad y = r - s,$$

where r and s are integers.

Substituting these values of x and y in (1) we have

$$2r^3 + 6rs^2 = 2t^3 - z^3.$$

This equation will be satisfied if we have simultaneously $r = t$, $z = -6rs^2$. If we write r in the form $r = k\rho^3$ where k has no cube factor except unity, then we must have $s = \pm k\sigma^3$. It is obvious that in the resulting solution the numbers x, y, z, t have the common factor k; and hence, for our present case, we have $k = 1$. Our desired solution of (1) then is

$$x = \rho^3 \pm 6\sigma^3, \; y = \rho^3 \pm 6\sigma^3, \; z = -6\rho\sigma^2, \; t = \rho^3. \tag{2}$$

It should be noticed that there is nothing in the argument to indicate whether (2) affords the general solution of (1). A similar remark applies to the other problems treated in this note.

It may be observed in passing that we have incidentally obtained a two-

[1] Presented to the American Mathematical Society, March 21, 1913.

[2] A single solution of this equation is given in the French Encyclopédie, tome I, vol. 3, p. 210; namely, $x = 7$, $y = 11$, $z = 102$, $t = 81$. This is not a special case of the solution obtained in the text.

parameter solution of the equation

$$x^3 + y^3 + z^3 = 2t^9,$$

x, y, z having the same values as before and t being equal to ρ. By taking ρ equal to the kth power of an integer we have further a two-parameter solution of the equation

$$x^3 + y^3 + z^3 = 2t^{9k}.$$

3. We shall now obtain solutions of the equation

$$x^3 + y^3 + z^3 + u^3 = 3t^3. \tag{3}$$

If we write

$$x = a + b - c, \quad y = a - b + c, \quad z = -a + b + c,$$

we have

$$a = \tfrac{1}{2}(x + y), \quad b = \tfrac{1}{2}(x + z), \quad c = \tfrac{1}{2}(y + z),$$

so that a number of the set a, b, c either is an integer or is the half of an odd integer. Substituting these values of x, y, z in (3) we obtain

$$(a + b + c)^3 - 24\, abc + u^3 = 3t^3.$$

This equation will be satisfied if we have simultaneously $u = -(a + b + c)$, $t^3 = -8abc$. If we write $a = m\alpha^3$, $b = n\beta^3$, $mn = ks^3$, where neither m nor n nor k has a cubic factor greater than 1, we have $c = \pm k^2\gamma^3$. Therefore we have the following five-parameter solution of (3)

$$x = m\alpha^3 + n\beta^3 \pm k^2\gamma^3, \quad y = m\alpha^3 - n\beta^3 \pm k^2\gamma^3, \quad z = -m\alpha^3 + n\beta^3 \pm k^2\gamma^3, \tag{4}$$

$$u = -(m\alpha^3 + n\beta^3 \pm k^2\gamma^3), \quad t = \pm 2ks\alpha\beta\gamma,$$

where $mn = ks^3$. The parameters entering into the solution represent either integers or the halves of odd integers, the latter being so related that the values of x, y, z, u, t are all integers.

It is clear that we may also readily obtain a five-parameter solution of the equation

$$x^3 + y^3 + z^3 + u^3 = kt^\alpha,$$

where α is any positive odd integer and k is any integer.

4. In the equation

$$x^3 + 2y^3 + 3z^3 = t^3, \tag{5}$$

let us make the substitution

$$x = r - s, \quad t = r + s;$$

then we have $2y^3 + 3z^3 = 2s^3 + 6r^2s$. This will be satisfied if we have simultaneously $y = s$, $z^3 = 2r^2s$. We exclude the case when r and s have a common factor greater than unity, since this would lead to a solution of x, y, z, t with such a common factor. Then it is clear that we have $s = m^3$, $r = 2n^3$. Hence we obtain the two-parameter solution

$$x = 2n^3 - m^3, \quad y = m^3, \quad z = 2mn^2, \quad t = 2n^3 + m^3. \tag{6}$$

It is easy to see that we have obtained incidentally a two-parameter solution of the equation

(7) $$x^3 + 2y^{3\alpha} + 3z^{3\beta} = t^3,$$

where α and β are any odd positive integers. For, let μ be the least common multiple of α and β and write

$$m = r^{\mu}, \quad 2n^2 = s^{\mu} = (2\cdot\sigma^2)^{\mu},$$

whence $n = 2^{\frac{\mu-1}{2}}\sigma^{\mu}$. These values of m and n, taken in connection with (6), lead to the following solution of (7):

(8) $$x = 2^{\frac{3\mu-1}{2}}\sigma^{3\mu} - r^{3\mu}, \quad y = r^{\frac{3\mu}{\alpha}}, \quad z = 2^{\frac{\mu}{\beta}}\sigma^{\frac{2\mu}{\beta}}r^{\frac{\mu}{\beta}}, \quad t = 2^{\frac{3\mu-1}{2}}\sigma^{3\mu} + r^{3\mu}.$$

As an analogue of equation (5) we have the equation

$$x^2 + 2y^2 = z^2$$

with the solution

$$x = 2t^2 - s^2, \quad y = 2st, \quad z = 2t^2 + s^2.$$

5. If for the equation

(9) $$x^4 + y^4 + 4z^4 = t^4$$

we confine attention to the case in which x, y, z, t have the greatest common divisor 1, it is easy to see that t is odd and either x or y is odd. We shall suppose that x is odd. Then if we write

$$x = r - s, \quad t = r + s,$$

it follows that r and s are integers. Substituting these values in (9) we have

$$y^4 + 4z^4 = 8r^3s + 8rs^3.$$

This equation will be satisfied if we have simultaneously $y^4 = 8r^3s$, $4z^4 = 8rs^3$ or $y^4 = 8rs^3$, $4z^4 = 8r^3s$. It is clear that these two cases will not lead to different solutions of (9).

Let us consider the first case: $y^4 = 8r^3s$, $4z^4 = 8rs^3$. If $r = k\rho^4$, where k has no fourth power factor except unity, then $s = 2k\sigma^4$, as we see from the first equation. Hence $4z^4 = 64k^4\rho^4\sigma^{12}$. If x, y, z, t are to have no common factor (except unity) it is clear that $k = 1$. Hence we are led to the following solution of (9):

(10) $$x = \rho^4 - 2\sigma^4, \quad y = 2\rho^3\sigma, \quad z = 2\rho\sigma^3, \quad t = \rho^4 + 2\sigma^4.$$

6. Proceeding in a similar way, we obtain for the equations

$$\begin{aligned} x^4 + 2y^4 + 2z^4 &= t^4, \\ x^4 + 8y^4 + 8z^4 &= t^4, \\ x^4 + y^4 + 2z^4 &= 2t^4, \\ x^4 + y^4 + 8z^4 &= 8t^4, \end{aligned}$$

the solutions

$$\begin{array}{llll} x = 4\rho^4 - \sigma^4, & y = 4\rho^3\sigma, & z = 2\rho\sigma^3, & t = 4\rho^4 + \sigma^4, \\ x = \rho^4 - \sigma^4, & y = \rho^3\sigma, & z = \rho\sigma^3, & t = \rho^4 + \sigma^4, \\ x = 2\rho^3\sigma, & y = 2\rho\sigma^3, & z = \rho^4 - \sigma^4, & t = \rho^4 + \sigma^4, \\ x = 8\rho^3\sigma, & y = 4\rho\sigma^3, & z = 4\rho^4 - \sigma^4, & t = 4\rho^4 + \sigma^4, \end{array}$$

respectively.

7. Let us consider next the equation

$$x^4 + y^4 + z^4 = 2t^4. \tag{11}$$

We shall seek solutions of this equation in which x, y, z, t are polynomials in an integral parameter k, the degree being not greater than 2. It is easy to see that the coefficients of k^2 in these polynomials must satisfy equation (11). The corresponding statement is true also of the independent terms. Now (11) has the obvious solutions obtained by putting one of the variables in the first member equal to zero and giving to the other variables in any way the values $+ 1$ or $- 1$. This gives us a certain number of cases to examine in order to see whether the coefficients of powers of k in the several polynomials can be so determined that (11) becomes an identity on substituting for the variables the polynomials in k.

Thus, as one case, we have to examine whether the ambiguous signs can be suitably chosen and the coefficients a, b, c, d so determined that

$$(k^2 + ak)^4 + (bk \pm 1)^4 + (k^2 + ck \pm 1)^4 = 2(k^2 + dk \pm 1)^4$$

is an algebraic identity in k. Carrying out the necessary reckoning in each of the possible cases which may arise we have the following solution of (11):

$$x = k^2 - 2k, \quad y = 2k - 1, \quad z = k^2 - 1, \quad t = k^2 - k + 1. \tag{12}$$

The reckoning is facilitated by observing that when k is replaced by $k + 1$ the resulting independent terms must still form a solution of (11).

THE CUBE ROOT OF A BINOMIAL SURD.

By ARTHUR C. JOHNSON, Hopedale, Mass.

In Wells's *Text Book in Algebra*, page 548, the equation $x^3 + x - 2 = 0$ leads by Cardan's method to the solution $\frac{1}{3}(\sqrt[3]{27 + 6\sqrt{21}} + \sqrt[3]{27 - 6\sqrt{21}})$, and the author implies that this cannot be simplified, for he says "there is no method in algebra for finding the cube root of a binomial surd."

The challenge suggested by this problem led to the following discussion, which, so far as the writer can learn, has not been published heretofore. Namely, to determine whether a quadratic binomial surd has a cube root which can be obtained by algebraic methods; and, when possible, to obtain this root.

Given the quadratic binomial surd $a + b\sqrt{c}$ in which a, b and c are positive

and rational, and $\sqrt{c}$ is a positive quadratic surd. We assume that the cube root of this expression, if obtainable, is a binomial of which one term x is positive and rational; the other term y is a positive quadratic surd. Thus

$$x + y = \sqrt[3]{a + b\sqrt{c}} \tag{1}$$

Then

$$x - y = \sqrt[3]{a - b\sqrt{c}}, \tag{2}$$

by the method shown in Section 269 of Wells's text. Hence

$$x^2 - y^2 = \sqrt[3]{a^2 - b^2c}. \tag{3}$$

Now since $x^2 - y^2$ is rational, the expression $\sqrt[3]{a^2 - b^2c}$ must also be rational. Consequently when the expression $a^2 - b^2c$ is a perfect cube, it will be possible to use this method.

Cubing (1),

$$x^3 + 3x^2y + 3xy^2 + y^3 = a + b\sqrt{c}. \tag{4}$$

Now the second and fourth terms of (4), in which odd powers of y occur, must be irrational, and the first and third terms must be rational.

Hence

$$3x^2y + y^3 = b\sqrt{c} \tag{5}$$

and

$$x^3 + 3xy^2 = a. \tag{6}$$

Combining (3) and (6), we have

$$4x^3 - 3x\sqrt[3]{a^2 - b^2c} - a = 0. \tag{7}$$

When $\sqrt[3]{a^2 - b^2c}$ is rational, (7) can be solved by the ordinary methods for cubic equations and the corresponding value of y can be found.

Whenever (7) has a commensurable root it is therefore possible to find the cube root of the given binomial surd by algebraic methods in the form of a quadratic binomial surd.

In the example quoted above we have to find the cube root of $27 + 6\sqrt{21}$.
In this case, $a = 27$, $b = 6$, $c = 21$.

$$\sqrt[3]{a^2 - b^2c} = \sqrt[3]{729 - 756} = \sqrt[3]{-27} = -3.$$

Substituting this value in (7), the equation becomes

$$4x^3 + 9x - 27 = 0,$$

and

$$x = \frac{3}{2}, \quad y = \frac{\sqrt{21}}{2}.$$

Hence the cube root of $27 + 6\sqrt{21}$ is $\frac{3}{2} + \frac{\sqrt{21}}{2}$, and in like manner the cube root of $27 - 6\sqrt{21}$ is $\frac{3}{2} - \frac{\sqrt{21}}{2}$.

Consequently

$$\frac{\sqrt[3]{27+6\sqrt{21}}+\sqrt[3]{27-6\sqrt{21}}}{3}=1,$$

the result expected from the equation $x^3+x-2=0$.

The method may be extended, and holds when a, b and c are negative as well as positive. Likewise the proof holds when the required root can be found in the form $a\sqrt{b}+c\sqrt{d}$.

The latter expression gives rise to the cubic equation

$$4bx^3-3x\sqrt[3]{a^2b-c^2d}-a=0. \qquad (8)$$

Example 1. To find the cube root of $2{,}261\sqrt{2}-1{,}845\sqrt{3}$. Here $a=2{,}261$, $b=2$, $c=-1{,}845$, $d=3$.

$$\sqrt[3]{a^2b-c^2d}=\sqrt[3]{10{,}224{,}242-10{,}212{,}075}=\sqrt[3]{12{,}167}=23.$$

Substituting this value in (8), the equation becomes

$$8x^3-69x-2{,}261=0,$$

and

$$x=7, \quad y=5.$$

Therefore the cube root of $2{,}261\sqrt{2}-1{,}845\sqrt{3}$ is $7\sqrt{2}-5\sqrt{3}$.

Example 2. To find the cube root of $276\sqrt{3}-70\sqrt{-2}$. Here $a=276$, $b=3$, $c=-70$, $d=-2$.

$$\sqrt[3]{a^2b-c^2d}=\sqrt[3]{228{,}528+9{,}800}=\sqrt[3]{238{,}328}=62.$$

Substituting this value in (8), the equation becomes

$$12x^3-186x-276=0,$$

whence $x=-2$, and $y=5$. Consequently the cube root of $276\sqrt{3}-70\sqrt{-2}$ is $-2\sqrt{3}+5\sqrt{-2}$.

BOOK REVIEWS.

W. H. Bussey, Chairman of the Committee.

Trigonometry for Schools and Colleges. By Frederick Anderegg and Edward Drake Roe. Revised edition by Frederick Anderegg. Ginn and Company, Boston, 1913. ix + 108 pages. 75 cents.

The characteristics which insured for this little book a favorable reception in its first edition, namely clearness and conciseness, have been retained in this new edition.

The striking alterations effected in the revision of the plane trigonometry touch the solution of triangles. The right triangle is solved in the third chapter instead of in the sixth and model solutions of both right and oblique triangles

are added. The discussion of the various cases has been somewhat elaborated and the number of numerical exercises increased.

In the more theoretical part of the text there are many minor changes in the direction of brevity, which completely compensate for the additions and leave the number of pages unaltered.

There are no changes in that part of the book which deals with spherical trigonometry.

J. W. BRADSHAW.

An Introduction to the Mathematical Theory of Heat Conduction. By L. R. INGERSOLL and O. J. ZOBEL. Ginn and Co., Boston, 1913. 171 pages.

This is a handy little volume on the Fourier theory, intended partly as an introduction to the general methods of mathematical physics, or rather of those parts which are concerned with boundary-value problems in the integration of partial differential equations, partly to give an account of a number of special problems selected from the wide range of applications in physics, engineering, and geology, as the sub-title suggests. In view of this two-fold aim it is convenient to consider separately the treatment of the general theory and of the special problems.

As one would expect, the general mathematical material introduced coincides in the main with that contained in the more elementary portions of Fourier's own work, with more modern notation and some simplification of arrangement. It covers the more important of those cases of steady and variable distribution of temperature which are solvable by simple formulæ in elementary functions, by ordinary trigonometric series, or by Fourier's integral, either directly or by the device of extension of the realm of definition of a function such as to make certain boundary properties automatic. The problem of the sphere is also included, as solved for the case of spherical isothermals by generalized Fourier series.

In the more purely mathematical portions of the book there are a number of things which seem to call for criticism, a few of which may be named here. Certain infinite series used in leading up to Fourier's integral are not convergent, and the integral itself is written in some places so that the inner integral diverges. In the explanation of the formula of conduction the terms "plane" and "lamina" seem to be used as synonymous. In one place it is stated that the assumption of expansibility of a function into a sine-series is to be considered justified if values can be found for the coefficients. The term "general solution," from which special solutions are to be obtained by "substituting," is used, without any real need, in connection with partial differential equations, where the notion is practically useless and usually disagreeably vague unless taken in the unenlightening sense of the class of all solutions; then the comforting assurance is added that many cases can be solved by other methods, even when it would be "almost impossible" to obtain the general solution of the differential equation. These few instances illustrate an occasional looseness of expression which one could

fairly be expected to avoid without sacrificing the desired condensation of statement to which it seems to be due.

The physical basis of Fourier's fundamental equations is also treated concisely, but space is found for a sufficient discussion of numerical computations and of the various units of measure used. The dimensions assigned to quantity of heat are questionable. In one place it is suggested that conduction in the gross is really an aggregate effect of transfers of heat between molecules, which may occur with the velocity of sound; one wonders what basis there is for this. As in the mathematical portions, such purely incidental and probably misleading remarks, too brief to be both accurate and intelligible, might better have been omitted. Also it is stated that conduction depends only on differences of temperature, the actual temperature being immaterial, then, further on, data are given showing the variation of conductivity with temperature, and in another place it is stated that the use of the mean conductivity for the range of temperature in question gives an approximate solution which may be practically close enough. Here a little more coherence in the arrangement might have made this fundamental matter clearer, and also have served to give needed emphasis to the limitations of Fourier's assumptions.

The chief merit of the book will probably be found in the special problems which are presented as illustrations of the practical bearing and value of the analytic formulas. It is apparent that the main labor of the authors has been devoted to collecting, sifting, and by some novelties, adding to this remarkable list of special applications, whose titles fill a column in the index. They are all of some intrinsic interest, and are worked out briefly, but still far enough to secure some numerically definite conclusions. The extent and variety of this material make the book unique among the works on this subject known to the reviewer. Mention should also be made of the many splendid diagrams, illustrating the convergence of Fourier series, or giving various temperature-curves.

A. C. LUNN.

PROBLEMS AND SOLUTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

Special Notice.—Please re-read the requests as to form of solutions on pp. 258–259 of the October issue. Unless these directions are observed by contributors, solutions must either be entirely rewritten by the committee or else rejected.

MANAGING EDITOR.

ALGEBRA.

397. Proposed by W. H. BUSSEY, University of Minnesota.

12 oxen are turned into a pasture of $3\frac{1}{3}$ acres and eat all the grass in 4 weeks so that the pasture is bare. 21 oxen are turned into a pasture of 10 acres and eat all the grass in 9 weeks. How many oxen will eat all the grass in a pasture of 24 acres in just exactly 18 weeks, it being assumed that the grass in all the pastures is at the same height when the oxen are turned in, and that the grass grows at a uniform rate?

GEOMETRY.

426. Proposed by R. D. CARMICHAEL, Indiana University.

On a given chord of a circle as a base construct an isosceles triangle, with vertex outside of the circle, such that its sides shall be divided in a given ratio by their points of intersection with the circle.

CALCULUS.

347. Proposed by R. D. CARMICHAEL, Indiana University.

Show that the differential equation

$$9\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5}-45\frac{d^2y}{dx^2}\frac{d^3y}{dx^3}\frac{d^4y}{dx^4}+40\left(\frac{d^3y}{dx^3}\right)^3=0$$

remains unchanged when the variables x and y undergo any projective transformation (Goursat-Hedrick, *Mathematical Analysis*, p. 86, ex. 18).

348. Proposed by E. L. DODD, University of Texas.

Let $(x_1, x_2, \cdots x_n)$ be a point in n dimensions lying in the "sphere" S defined by

$$x_1^2+x_2^2+\cdots+x_n^2\leqq 1.$$

Let T be that part of S defined by a set of n linear homogeneous inequalities with non-vanishing determinant; thus:

$$a_ix_1+b_ix_2+\cdots+k_ix_n\geqq 0,\quad i=1, 2, \cdots n.$$

Find the value of

$$\frac{\int\cdot\underset{T}{\cdot}\cdot\int dx_1\cdots dx_n}{\int\cdot\underset{S}{\cdot}\cdot\int dx_1\cdots dx_n};$$

in other words, find the magnitude of a "solid angle" in n dimensions, with the "sphere" as unit solid angle.

Note.—This problem was discussed and left unsolved by Schläfli in the Quarterly Journal of Mathematics for 1858, 1860, 1867. EDITOR.

349. Proposed by C. N. SCHMALL, New York City.

If $y=a\cos(\log x)+b\sin(\log x)$, eliminate the constants a and b and obtain the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0.$$

MECHANICS.

282. Proposed by R. P. LOCHNER, Philadelphia, Pa.

A car weighing 10 tons (2,240 lb. each) attains a speed of 15 miles per hour from rest in 24 seconds, during which it covers 100 yards. If the space-average of the resistances is 30 lb. per ton, find the average horse-power used to drive the car. (Morley's "*Mechanics for Engineers,*" p. 66).

283. Proposed by C. N. SCHMALL, New York, N. Y.

The maximum length of a certain chain which can be suspended from one end without breaking is l. It is desired to form a catenary with a length $2l/k$ of the chain, the points of support being a distance d apart, in the same horizontal line.

Show that the maximum value of d is $\frac{2l}{k}(k^2-1)^{\frac{1}{2}}\log\left(\frac{k+1}{k-1}\right)^{\frac{1}{2}}$.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

199. Proposed by R. P. LOCHNER, Philadelphia, Pa.

Find three integral squares such that the sum of every two of them shall be a square.—Alsop's "Algebra."

200. Proposed by R. D. CARMICHAEL, Indiana University.

Find the general solution, in relatively prime integers, of the equation $x^2 + y^2 = z^4$.

201. Proposed by E. T. BELL, Seattle, Washington.

Eisenstein (*Crelle*, t. 27, p. 282) proposed, as the simplest of several problems: "In the expansion of

$$\frac{1 + z + z^2 + \cdots + z^{p-1}}{(1 - z)^{p-1}} - 1,$$

where p is prime, to show that the coefficients of the various powers of z are all divisible by p."

202. Proposed by A. R. SCHWEITZER, Chicago, Ill.

There exists an infinitude of systems of dyads $\{\alpha\beta\}$ in 7, 9, 11, etc., elements such that each system has the following properties: (1) if $\alpha\beta$ is in the set, then $\beta\alpha$ is not in the set; (2) for each dyad $\alpha\beta$ in the set there exists an element ξ such that $\xi\beta$ and $\alpha\xi$ are also in the set. For example, such a system is,

12,	23,	34,	45,	56,	67,	78,	89,	91
13,	24,	35,	46,	57,	68,	79,	81,	92
14,	25,	36,	47,	58,	69,	71,	82,	93
51,	62,	73,	84,	95,	16,	27,	38,	49

Investigate the existence of

I. A finite set of triads $\{\alpha\beta\gamma\}$ such that (1) if $\alpha\beta\gamma$ is in the set, then $\beta\gamma\alpha$, $\gamma\alpha\beta$ are also in the set but $\beta\alpha\gamma$ is not in the set, (2) for each triad $\alpha\beta\gamma$ in the set there exists an element ξ such that $\xi\beta\gamma$, $\alpha\xi\gamma$, $\alpha\beta\xi$ are also in the set.

II. A finite set of tetrads $\{\alpha\beta\gamma\delta\}$ such that (1) if $\alpha\beta\gamma\delta$ is in the set, then $\beta\gamma\alpha\delta$, $\gamma\alpha\beta\delta$, $\gamma\delta\alpha\beta$ are also in the set but $\beta\alpha\gamma\delta$ is not in the set, (2) for each tetrad $\alpha\beta\gamma\delta$ in the set there exists an element ξ such that $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$ are also in the set.

The problem for alternating *n-ads* for $n > 4$ is obvious.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

382. Proposed by C. E. FLANAGAN, Wheeling, W. Va.

A few days ago, I deduced the following formula for finding the value of the unknown quantity in a cubic equation having the form, $x^3 + 3A^2x = B$.

Let $C = \sqrt{\dfrac{2B}{A^3} + 1} - 1$. Then $x = \dfrac{2B}{A^2(C^2 + 12)} + \dfrac{AC}{4}$.

It is required to show: (1) How this formula was derived; (2) under what conditions does it give exact results; (3) in general, what is its degree of approximation; (4) if possible, modify it so that it will always give exact results; (5) if it cannot be so modified, show why.

SOLUTION BY THE PROPOSER.

(1) The formula was derived by equating the volume of a spherical segment of one base and unknown altitude, x, to n times the volume of a sphere whose radius is the radius of the base of the spherical segment.

Thus, let r be the given radius of the base of the spherical segment of one base and x the unknown altitude. Then we have by hypothesis

$$\tfrac{1}{3}\pi x^2(3r - x) = n(\tfrac{4}{3}\pi r^3) = \tfrac{1}{2}\pi x r^2 + \tfrac{1}{6}\pi x^3.$$

Hence,

$$3rx^2 - x^3 = 4nr^3,$$

and

$$3r^2x + x^3 = 8nr^3;$$

whence, by adding the two equations and reducing,

$$r^2x + rx^2 = 4nr^3.$$

Whence

$$x = -\frac{r}{2} \pm \frac{r}{2}\sqrt{16n+1}.$$

Letting $r^2 = A^2$, $B = 8nr^3$, and $C = \sqrt{\frac{2B}{A^3}+1} - 1$, we have

$$x = \frac{AC}{2}. \tag{1}$$

But

$$\frac{B}{\dfrac{A^3C^3}{8}+\dfrac{3A^3C}{2}} = 1.$$

Hence,

$$x = \frac{AC}{2} \times \frac{B}{\dfrac{A^3C^3}{8}+\dfrac{3A^3C}{2}} = \frac{4B}{A^2(C^2+12)}. \tag{2}$$

From the last two values of x, we have, by addition and dividing by 2,

$$x = \frac{2B}{A^2(C^2+12)} + \frac{AC}{4}. \tag{3}$$

(2) This always gives exact values when $n = \frac{1}{2}$.

As an application, take the cubic equation,

$$x^3 + 12x = 1{,}120.$$

Here $A = 2$, $B = 1{,}120$, and $C = \sqrt{281} - 1$. Hence, from (1), $x = 15.7$; from (2), $x = 8.4$, and from (3), $x = 10.03$. Thus (3) gives a very close approximation.

Remark by the Editors.

The proposer does not answer (3), (4), and (5). The deductions in his solution result from the confusion of letters representing different quantities and using them to represent the same quantity without drawing the necessary conclusion therefrom. Thus, in the expression $\frac{1}{3}\pi x^2 \times (3r - x)$, the r here is the radius of the sphere from which the segment is taken, whereas in $\frac{1}{2}\pi r^2 x + \frac{1}{6}\pi x^3$, the r is the radius of the base of the segment. If these two expressions are equated as they are in the solution above, it follows that $x = r$ and therefore $n = \frac{1}{2}$. It thus follows that when $n = \frac{1}{2}$ the formula gives correct values for cubics of the proposed form. This seems to us to answer the remaining questions in his problem.

387. Proposed by H. C. FEEMSTER, York College, Nebraska.

Sum the following series:

$$(1)\quad \sum_{n=1}^{n=\infty} \frac{1}{8^{n+1}n!};\quad (2)\ \sum_{n=1}^{n=\infty} \frac{1\cdot 3\cdot 5\cdots 2n+1}{2^{2n+1}(2n-1)!};\quad (3)\ \sum_{n=1}^{n=a} \frac{1}{2^{2n}n!}.$$

SOLUTION BY J. A. BULLARD, Worcester Polytechnic Institute.

By Maclaurin's Theorem, we have

$$f(x) = \sum_{n=0}^{n=\infty} \frac{x^n}{n!} D^n f(0), \tag{A}$$

where $D^n f(0) = [D^n f(x)]_{x=0}$, $0! = 1$ and $D^0 f(0) = f(0)$.

(1) Let $f(x) = e^x$. Then by (A),

$$\frac{e^x}{8} = \frac{1}{8}\sum_{n=0}^{n=\infty}\frac{x^n}{n!} = \frac{1}{8}\left\{1 + \sum_{n=1}^{n=\infty}\frac{x^n}{n!}\right\}. \qquad (B)$$

Letting $x = \frac{1}{8}$, and reducing, we readily have

$$\frac{e^{\frac{1}{8}} - 1}{8} = \sum_{n=1}^{n=\infty}\frac{1}{8^{n+1}n!}.$$

(2) $$\sum_{n=1}^{n=\infty}\frac{1\cdot 3\cdot 5\cdots 2n+1}{2^{2n+1}(2n-1)!} = \sum_{n=1}^{n=\infty}\frac{(2n+1)!}{2^{3n+1}(2n-1)!\,n!}$$

$$= \sum_{n=1}^{n=\infty}\frac{2n+1}{2^{3n}(n-1)!} = \frac{1}{8}\sum_{n=1}^{n=\infty}\frac{2n+1}{8^{n-1}(n-1)!}.$$

Let $f(x) = (2x+3)e^x$. Then $D^0f(0) = 3$, $D'f(0) = 5$, $\cdots$, $D^nf(0) = 2n+3$. By equation (A), we have

$$\frac{(2x+3)e^x}{8} = \frac{1}{8}\sum_{n=0}^{n=\infty}\frac{x^n}{n!}(2n+3).$$

Letting $x = \frac{1}{8}$, we have

$$\frac{(\frac{1}{4}+3)e^{\frac{1}{8}}}{8} = \frac{1}{8}\sum_{n=0}^{n=\infty}\frac{2n+3}{8^n n!}.$$

Replacing n by $n-1$, we have

$$\frac{13e^{\frac{1}{8}}}{32} = \frac{1}{8}\sum_{n=1}^{n=\infty}\frac{2n+1}{8^{n-1}(n-1)!} = \sum_{n=1}^{n=\infty}\frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{2n+1}(2n-1)!}.$$

(3) Letting $x = \dfrac{1}{2^2}$ in equation (B) and multiplying by 8, we have

$$e^{\frac{1}{4}} - 1 = \sum_{n=1}^{n=\infty}\frac{1}{2^{2n}n!}.$$

Excellent solutions were received from A. M. Harding, R. M. Mathews, G. N. Bauer, H. L. Slobin, and S. Lefschetz.

Note.—The Proposer intended the sign in the denominator of (2) to be + instead of −. With that change his answer for (2) is $\frac{1}{2} - \frac{1}{3}\sqrt{2}$. Editor.

388. Proposed by JOSEPH V. COLLINS, Stevens Point, Wis.

On a certain typewriter there is a double scale as follows:

80··· 70···60···50···40···30···20···10··· 0
1··· 5···10···15···20···25···30···35···40

It is used to locate headings in the middle of the page. To space a heading, one sets the machine at the right stop and with the spacer counts out the number of letters and spaces in the heading. To the reading on the 40 scale where the carriage stops is added the reading of the right stop on the same scale. This number is the one on which to set the carriage pointer on the 80 scale to begin the heading. Show by algebra that the method is correct.

SOLUTION BY THE PROPOSER.

Let a = number of letters and spaces in heading.
b = number of spaces on 80 scale of right stop.

Then $\frac{80-a}{2}$ = number on 80 scale to begin heading. But by the method used this number is found as $\left\{\left(40-\frac{b}{2}\right)-\frac{a}{2}\right\}+\frac{b}{2}$.

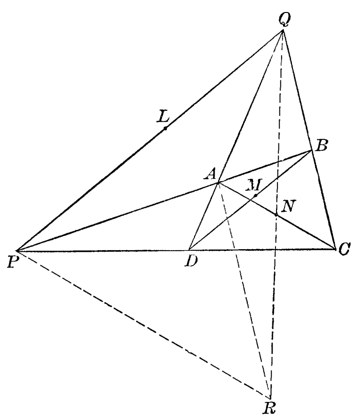
GEOMETRY.

418. Proposed by MRS. H. E. TREFETHEN, Waterville, Maine.

The difference of the squares of the two interior diagonals of a cyclic quadrilateral is to twice their rectangle as the distance between their mid-points is to the third diagonal.

SOLUTION BY C. N. SCHMALL, New York, N. Y.

Let $ABCD$ be a cyclic quadrilateral. Let the sides BA, CD meet in P, and the sides DA, CB in Q. Draw PQ. The resulting figure is a *complete quadri-*



lateral and PQ is the *third diagonal.* Let L, M, N, be the mid-points of the three diagonals. Then we have to prove that

$$\frac{\overline{AC}^2-\overline{BD}^2}{2AC\cdot BD}=\frac{MN}{PQ}.$$

It is well known that the points, L, M, N, are collinear. (See RUSSELL'S *Sequel to Elem. Geom.*, p. 154, ex. 5; or LACHLAN'S *Pure Geom.*, p. 93, § 151; or MILNE'S *Cross-Ratio Geom.*, p. 113, ex. 3.) Hence $MN = LN - LM$, or

$$\frac{MN}{PQ}=\frac{LN}{PQ}-\frac{LM}{PQ}. \tag{1}$$

Draw QN and prolong it to R, making $NR = QN$. Draw AR. Then $QCRA$

is a parallelogram. Hence QC is parallel to AR and

$$QC = AR. \tag{2}$$

Now, the triangles PAC, PDB, are similar; for $\angle PCA = \angle PBD$, both standing on the arc AD in a cyclic quadrilateral. Hence

$$AC : DB :: PA : PD. \tag{3}$$

Likewise, the triangles QCA, QDB are similar; and

$$AC : DB :: QC : QD. \tag{4}$$

From (3) and (4),

$$PA : PD :: QC : QD;$$

or

$$PD : QD :: PA : QC;$$

or, by (2),

$$PD : QD :: PA : AR. \tag{5}$$

Again,

$$\angle PDQ = \angle PAR, \tag{6}$$

for

$$\angle PDQ = 180° - \angle ADC = \angle ABC$$

and

$$\angle PAR = \angle ABC.$$

Hence, by (5) and (6) the triangles PDQ and PAR are similar; since they have an angle in each equal and the including sides proportional. Therefore,

$$PA : PD :: PR : PQ;$$

or

$$PA : PD :: 2LN : PQ. \tag{7}$$

By (3) and (7) we have

$$AC : BD :: 2LN : PQ. \tag{8}$$

In a similar way, by drawing QM and prolonging it to S (say), making $MS = QM$, and then drawing PS, we can show that

$$AC : BD :: PQ : 2LM. \tag{9}$$

From (8) and (9),

$$\frac{LN}{PQ} = \frac{1}{2}\frac{AC}{BD}, \qquad \frac{LM}{PQ} = \frac{1}{2}\frac{BD}{AC}.$$

Substituting these values in (1), we have

$$\frac{MN}{PQ} = \frac{1}{2}\left(\frac{AC}{BD} - \frac{BD}{AC}\right); \qquad \text{or} \qquad \frac{MN}{PQ} = \frac{\overline{AC}^2 - \overline{BD}^2}{2AC \cdot BD}.$$

CALCULUS.

333. Proposed by C. N. SCHMALL, New York City.

Evaluate $\int\int\int \sqrt{\frac{1 - (x^2 + y^2 + z^2)}{1 + x^2 + y^2 + z^2}}\, dx\, dy\, dz$, where $x^2 + y^2 + z^2 < 1$.

SOLUTION BY THE PROPOSER.

The given integral evidently depends on the distance of the point (x, y, z) from the origin. As the integration is to be taken for all values of x, y, and z within a sphere of unit radius, center at the origin, the integral may be put into the form

$$u = 4\pi \int_0^1 \sqrt{\frac{1 - r^2}{1 + r^2}} \cdot r^2 dr,$$

in which the element of volume is a thin spherical shell included between the spheres of radii r, $r + dr$.

Now put $r = \sin\phi$. Then we have

$$u = 4\pi \int_0^{\frac{\pi}{2}} \frac{\cos^2\phi \sin^2\phi}{\sqrt{1 + \sin^2\phi}} d\phi = 4\pi \int_0^{\frac{\pi}{2}} \frac{\frac{1}{4}\sin^2 2\phi}{\sqrt{1 + \sin^2\phi}} d\phi \tag{1}$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \left(\frac{\frac{1}{4}\sin^2 2\phi}{\sqrt{1 + \sin^2\phi}} + \cos 2\phi \sqrt{1 + \sin^2\phi} \right) d\phi - 4\pi \int_0^{\frac{\pi}{2}} \cos 2\phi \sqrt{1 + \sin^2\phi}\, d\phi$$

$$= 4\pi \Big[\tfrac{1}{2} \sin 2\phi \sqrt{1 + \sin^2\phi} \Big]_0^{\frac{\pi}{2}} - 4\pi \int_0^{\frac{\pi}{2}} \cos 2\phi \sqrt{1 + \sin^2\phi}\, d\phi$$

$$= -4\pi \int_0^{\frac{\pi}{2}} \cos 2\phi \sqrt{1 + \sin^2\phi}\, d\phi = 4\pi \int_0^{\frac{\pi}{2}} (2\sin^2\phi - 1)\sqrt{1 + \sin^2\phi}\, d\phi$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \frac{(1 - 2\cos^2\phi)(1 + \sin^2\phi)\, d\phi}{\sqrt{1 + \sin^2\phi}}$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \frac{1 - 2\cos^2\phi + \sin^2\phi - 2\sin^2\phi \cos^2\phi}{\sqrt{1 + \sin^2\phi}} d\phi$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \frac{1 - 2\cos^2\phi + \sin^2\phi}{\sqrt{1 + \sin^2\phi}} d\phi - 2u \quad \text{(by equation (1) above).}$$

Hence

$$3u = 4\pi \int_0^{\frac{\pi}{2}} \frac{3\sin^2\phi - 1}{\sqrt{1 + \sin^2\phi}} d\phi = 4\pi \int_0^{\frac{\pi}{2}} \left(3\sqrt{1 + \sin^2\phi} - \frac{4}{\sqrt{1 + \sin^2\phi}} \right) d\phi$$

$$= 4\pi \int_0^{\frac{\pi}{2}} 3\sqrt{1 + \sin^2\phi}\, d\phi - 4\pi \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 + \sin^2\phi}}.$$

Putting $\pi/2 - \phi$ for ϕ, we obtain

$$3u = 12\sqrt{2}\cdot\pi \int_0^{\frac{\pi}{2}} \sqrt{1 - \tfrac{1}{2}\sin^2\phi}\, d\phi - 16\sqrt{2}\cdot\pi \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}}.$$

These are complete elliptic integrals of the second and first kinds respectively, and can be readily evaluated. See Peirce's *Short Table of Integrals*, pp. 118–119, and Bromwich's *Infinite Series*, p. 162, ex. 6.

Using the notation employed in the former, we have

$$u = \tfrac{1}{3}\{12\pi\sqrt{2}E(\sqrt{\tfrac{1}{2}},\ \phi) - 16\pi\sqrt{2}F(\sqrt{\tfrac{1}{2}},\ \phi)\}.$$

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

188. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

Solve the congruence $3x^2 + 4x + 5 \equiv 0$ [mod 20].

SOLUTION BY WALTER C. EELLS, Whitworth College.

By inspection, solutions of the two congruences,

$$3x^2 + 4x + 5 \equiv 0 \pmod 4, \tag{1}$$

$$3x^2 + 4x + 5 \equiv 0 \pmod 5, \tag{2}$$

are $x = \pm 1$ for (1), and $x = 0, 2$ for (2).

Then we have the four systems of linear congruences,

$$(3)\ \begin{cases} x \equiv 1 \pmod 4, \\ x \equiv 0 \pmod 5, \end{cases} \qquad (4)\ \begin{cases} x \equiv 1 \pmod 4, \\ x \equiv 2 \pmod 5, \end{cases}$$

$$(5)\ \begin{cases} x \equiv -1 \pmod 4, \\ x \equiv 0 \pmod 5, \end{cases} \qquad (6)\ \begin{cases} x \equiv -1 \pmod 4, \\ x \equiv 2 \pmod 5. \end{cases}$$

To solve system (3), substitute $x = 1 + 4y$ from first congruence in second congruence,

$$1 + 4y \equiv 0 \pmod 5;$$

whence

$$y = 1 + 5k,$$

and

$$x = 1 + 4y = 1 + 4(1 + 5k) = 5 + 20k.$$

Hence, $x \equiv 5 \pmod{20}$, yielding one solution of given congruence. Similarly, systems (4), (5), (6) yield $x = 17, 15, 7$, giving altogether four independent roots of the congruence $3x^2 + 4x + 5 \equiv 0 \pmod{20}$.

Also solved by H. C. FEEMSTER, LOUIS CLARK, S. LEFSCHETZ, E. B. ESCOTT.

No solution of 189 has been received.

MISCELLANEOUS QUESTIONS.

Edited by R. D. Carmichael.

When it was necessary to make ready the manuscript for this issue our readers had not yet had time to reply to our first questions, proposed in the November number. It is believed that some replies to those questions may be inserted in the issue next following this one. In the meantime we have received two other interesting questions. They are printed here; and the attention of our readers is directed to them.

QUESTIONS.

3. In connection with the theory of the conduction of electricity through gases, one is led to the differential equation

$$y\frac{d^2y}{dx^2} + a\left(\frac{dy}{dx}\right)^2 + b\frac{dy}{dx} + cy + d = 0,$$

where a, b, c, d are constants. For unrestricted values of a, b, c, d the solution of this differential equation presents peculiar difficulties, the series solutions obtained by the customary methods having (apparently) too small a range of convergence to be satisfactory from the point of view of electrical theory. The general solution of this equation is wanted in case it can be found. If no general solution is obtained for unrestricted a, b, c, d, it is desirable to know special values of a, b, c, d or special relations among a, b, c, d which make it possible to find the general solution; and this solution is desired in each case.

We shall be glad to receive an answer to any part of this question in case a complete answer is not found.

4. In analytic geometry classes we teach about six formulas for the straight line and often we drill pupils in their use only with formal exercises. Mr. R. M. Mathews, of Riverside, California, proposes a collaboration through the Monthly by means of which a set of problems of interest in themselves shall be collected for these drill exercises. Our readers are requested to send to the editor of this department such problems, classified under headings suggesting the formula for which each problem is useful, the headings being as follows: two point form, slope point form, slope intercept form, two intercept form, normal form, general form.

As examples of the sort of problem desired we have the following, suggested by Mr. Mathews:

Two point form.—A linear relation connects degrees centigrade with degrees Fahrenheit. Find this relation if 50° F. = 10° C. and − 4° F. = − 20° C.

Slope intercept form.—The electrical resistance of annealed copper wire is 9.59 ohms per mil-foot at 0° C., from which point it increases with the temperature in the proportion of 403 per thousand. Express the resistance in terms of temperature centigrade.

NOTES AND NEWS.

Under the Direction of Florian Cajori.

The winter meeting of the Chicago Section of the American Mathematical Society wil held at Chicago on December 26–27, 1913.

The October number of the *Popular Science Monthly* contains an article on the Fourth Dimension, by Professor Samuel M. Barton, of the University of the South.

The firm of Gauthier-Villars is undertaking the publication of the complete works of the late Henri Poincaré, under the direction of the Paris Academy of Sciences and the French Minister of Public Instruction.

H. Grassmann, of the University of Giessen, the second volume of whose *Projektive Géometrie* has just been brought out by Teubner, is a son of the former professor in the gymnasium at Stettin, who is celebrated as the founder of the Ausdehnungslehre.

The San Francisco section of the American Mathematical Society held a regular meeting on October 25 at Stanford University. The following names appeared on the program: Professors L. M. Hoskins, M. W. Haskell, W. C. Eells, G. A. Miller, W. A. Manning and Mr. B. A. Bernstein.

Mr. A. C. Johnson, formerly of Cincinnati, Ohio, has recently become principal of the high school at Hopedale, Mass.

Dr. N. J. LENNES, of the department of mathematics at Columbia University, has accepted the position of professor of mathematics at the University of Montana, where he succeeds Professor L. C. Plant, who goes to the Michigan Agricultural College as head of the department of mathematics there.

Owing to the important rôle that questions of psychology play at the present time in discussions of mathematical teaching, a booklet on *Psychologie und mathematischer Unterricht,* prepared by D. Katz, under the editorship of Professor FELIX KLEIN, will be of general interest.

On September 6, 1913, occurred the death of Professor LUCIAN AUGUSTUS WAIT, of Cornell University, at the age of 67 years. After graduating at Harvard in 1870, he served on the mathematical faculty of Cornell until 1910, when he became professor emeritus. He is well known as one of the authors of the Oliver Wait & Jones series of mathematical texts.

Professor CHARLES GREENE ROCKWOOD, of Princeton, died on July 2, 1913, at the age of 71 years. He was a Yale graduate. He began his career as a teacher of college mathematics at Bowdoin and Rutgers, and during 1877–1905 was professor of mathematics at Princeton. Since 1905 he was professor emeritus. His early teaching embraced also natural philosophy and astronomy. His wide scientific interests are evident from the large number of scientific societies of which he was a member. He took part in the Princeton eclipse expedition to Colorado in 1878 and was a contributor on seismology to the *American Journal of Science.*

Recommendations on the unification of mathematical notations are contained in Heft 17 issued by the German committee on the teaching of mathematics and science. The recommendations are reprinted in Schotten's *Zeitschrift* for September, 1913. It will be remembered that a report on the same subject was made by an American committee under the chairmanship of L. P. Jocelyn, of Ann Arbor.

Dr. ALEXANDER MACFARLANE, whose death was announced in the last number of the MONTHLY, wrote to one of the editors less than two weeks before his death with regard to the history of the exponential and logarithmic concepts which appeared in the MONTHLY. He expressed himself as follows: "These concepts play the leading parts in the higher developments of vector-analysis; therefore

it behooves every vector-analyst to make himself thoroughly familiar with the history."

Plans are being made to commemorate the seventh centenary of Roger Bacon's birth at Oxford in July, 1914, when a statue of Bacon will be unveiled, and addresses will be given by distinguished scholars. There will be issued a memorial volume of essays dealing with various aspects of Roger Bacon's work. It is also proposed to prepare an edition of Bacon's writings, including various MSS. not hitherto published. To secure funds for such an edition it is proposed to organize a Roger Bacon Society.

At the University of Michigan the following men have been appointed as instructors in mathematics: T. Fort, Ph.D. (Harvard) 1912, student at Göttingen, 1912–1913; L. J. Rouse, A.B. (Princeton) 1908, Scholar at Univ. of Pennsylvania, 1912–1913; C. H. Forsythe, A.M. (Illinois) 1910, graduate student at Cornell, Illinois, and Michigan; B. Libby, B.S. (Princeton) 1911, graduate student at Wisconsin, 1911–1913. At the October meeting of the Board of Regents Assistant Professor T. R. Running was promoted to a junior professorship in mathematics and Dr. Clyde E. Love, from an instructorship to an assistant professorship. Assistant Professor Peter Field was promoted to a junior professorship at a previous meeting. Mr. L. A. Hopkins is absent on leave, as Fellow in the University of Chicago, and Mr. Carl Coe as Scholar at Harvard University.

According to *Science*, W. H. YOUNG, professor of mathematics in Liverpool University, has been appointed professor of mathematics in the University of Calcutta, for the purpose of organizing there a new school of mathematics. As the duties of the post require his residence in India only from November till March, he will retain his professorship in Liverpool University.

In the *Annaes da academia polytechnica do Porto* (Volume VIII, 1913) Yoshio Mikami gives an account of a seventeenth century Portuguese scientist who went to Japan as a Christian missionary, but, to escape persecution by the Shogunate, was compelled to forsake his faith and render service to the authorities. He assumed a Japanese name. His real name is still unknown. In 1650 he translated into Japanese an astronomical treatise. Mikami concludes that Japanese scientists of the seventeenth century became acquainted with European astronomy not so much through the Chinese, as is sometimes believed, but mainly through direct contact with Europeans. In 1687 the Shogunate prohibited the importation into Japan of 38 works compiled in China from European sources, for the reason that they bore on "the accursed religion of Christianity." Among these treatises were a translation of Euclid, a treatise on arithmetic, and some works in astronomy and surveying.

The Annual Report of the Smithsonian Institution for 1912, published in 1913, contains the address on "Molecular theories and mathematics," delivered by Professor ÉMIL BOREL, of Paris, on the occasion of the inauguration of the Rice Institute, October 10 to 12, 1912. This volume contains three other articles of special mathematical interest, as follows: "The connection between the ether and

matter," by HENRI POINCARÉ; "Modern mathematical research," by G. A. MILLER; and "HENRI POINCARÉ, his scientific work, his philosophy," by CHARLES NORDMANN.

The central committee of the International Commission on the Teaching of Mathematics announces a Congress of the Commission to be held at Paris, April 6 to 8, 1914. The details of the meeting will be announced later. The arrangements at Paris were in charge of Professor Carlo Bourlet, who died on the twelfth of August, from the effects of an accident, and consequently the completion of the program is delayed. At present, it can be definitely announced that two public sessions will be devoted to the consideration of reports on the following two questions:

A. The results obtained in the introduction of the differential and the integral calculus into secondary instruction. General Reporter, PROFESSOR E. BEKE, Budapest.

B. The place and function of mathematics in advanced technological instruction. General Reporter, PROFESSOR P. STAECKEL, Heidelberg.

These reports will seek to summarize conditions and tendencies as to these questions throughout the world, and will be based on replies received from the various nations in response to a questionnaire that has been prepared on each topic. It is planned, further, to hold a meeting of the Commission at Munich, in the summer of 1915, and a final meeting in connection with the Sixth International Congress of Mathematicians at Stockholm, in 1916.

It is to be noted that the term "secondary instruction," as used in the foregoing announcement, refers to instruction given in the last two years of schools like the German *gymnasia*, and would be comparable in the United States to the fifth and sixth year in a six-year high school or to the freshmen and sophomore courses in our colleges. The chief interest to us in this report will be its bearing upon the general question as to whether it would be desirable to teach the elements of the calculus much earlier than we do, for instance, in connection with college algebra and analytic geometry.

Dr. H. F. Baker's presidential address before the mathematical and physical science section of the British Association for the Advancement of Science, was entitled "The Place of Pure Mathematics." It is published in full in *Science* of September 12, 1913. In the course of the introduction he says: "What, then, is this subject? What can it be about if it is not primarily directed to the discussion of the laws of natural phenomena? What kind of things are they that can occupy alone the thoughts of a life-time? I propose now to attempt to answer this, most inadequately, by a bare recital of some of the broader issues of present interest."

Under "Precision of Definitions" Dr. Baker points out that "it is a constantly recurring need of science to reconsider the exact implication of the terms employed," and illustrates this by reference to "those series which Fourier used so boldly, and so wickedly, for the conduction of heat." "Like all discoverers, he took much for granted. Precisely how much is the problem. This problem

has led to the precision of what is meant by a function of real variables, to the question of the uniform convergence of an infinite series, as you may see in early papers of Stokes, to new formulation of the conditions of integration and of the properties of multiple integrals, and so on. And it remains still incompletely solved."

Dr. Baker then refers to the calculus of variations, and is led to speak of the stability of the solar system. "For those who can make pronouncements in regard to this I have a feeling of envy; for their methods, as yet, I have quite another feeling. The interest of this problem alone is sufficient to justify the craving of a pure mathematician for powerful methods and unexceptionable rigor." He then proceeds to philosophical considerations on non-euclidean geometry, the theory of groups, algebraic functions, functions of complex variables, and the theory of numbers. "Each of those I have named is large enough for one man's thought; but they are interwoven and interlaced in indissoluble fashion and form one mighty whole, so that to be ignorant of one is to be weaker in all. I am not concerned to depreciate other pursuits, which seem at first sight more practical; I wish only, indeed, as we all do, it were possible for one man to cover the whole field of scientific research; and I vigorously resent the suggestion that those who follow these studies are less careful than others of the urgent needs of our national life."

The "southerly deviation of falling bodies" has recently been re-examined mathematically by three Americans. Dr. W. H. Roever, of Washington University, has contributed articles to the *Transactions* of the Amer. Math. Soc. (Vol. 12, 1911, 335–353; Vol. 13, 1912, 469–490), in which under the assumption of a distribution of revolution; that is, assuming the potential function of the earth's gravitational field of force to be of the form $f(r, z)$, where r is the distance of a general point from the earth's axis of rotation and z is that from a fixed plane perpendicular to the axis, he obtains a formula for the southerly deviation which reduces, when g is taken constant, to the formula given by Gauss. Roever showed also that local irregularities in the earth's gravitational field of force caused by mountains and mineral deposits may influence the southerly deviation to the extent of from -16 times to $+40$ times the amount which is given by the formula of Gauss for the southerly deviation.

The theory of the problem of the southerly deviation is reconsidered by President R. S. Woodward of the Carnegie Institution (*Astronomical Journal*, Vol. 28, 1913, 17–29; *Science*, N. S., Vol. 38, 1913, 315–319), who asserts that the problem has "hitherto been inadequately treated by reason of neglect of the effect of the square of the angular velocity of the earth and other terms of the same order, and by equal neglect of the effect of difference between the geocentric latitude of the point of departure of the body and the geographical latitude of the horizontal plane to which the body falls. Failure to consider these effects has led to erroneous conclusions in respect to the meridional deviation of the body, the real deviation being, in fact, opposite to that derived by Gauss and most later investigators." According to Woodward's deductions, "for a fall of

10 seconds, or 490.24 meters (in vacuo) in latitude 45°, the meridional deviation would be 3.03 cm. and the easterly deviation 16.85 cm." Most experiments carried on in the northern hemisphere have shown a very slight deviation, not to the north, as Woodward's formula demands, but toward the south. Says Woodward, "I consider it quite impractical to make any conclusive experiments on the deviation of spheres falling in air."

A third treatment of freely falling bodies was presented on October 25, 1913, by Professor L. M. HOSKINS, of Stanford University, before the San Francisco section of the American Mathematical Society. According to the abstract of the paper, Professor Hoskins referred the motion, as one of the simplest methods of studying the problem, "to axes fixed in the earth, making use of the relation between the accelerations of a particle referred to fixed and moving axes. . . . The question whether this field would cause a deviation northward or southward from a plumb-line suspended from the point where the body starts, is closely connected with the question of the curvature of the lines of force in this field; the path would lie on the convex side of that line of force passing through the position of rest, while the plumb-line would hang on the concave side. So far as the effect of this field is concerned, the body will fall on that side of the plumb-line toward which the lines of force are convex. Analysis indicates that the convexity is toward the equator."

This issue brings to a close the first year of the MONTHLY under its new auspices. The question whether a journal devoted primarily to the collegiate field of mathematics would be appreciated and supported has been answered in the affirmative to a gratifying extent. The subscription list has increased continuously throughout the year, the acceleration being greatest during the past three months. There seems to be every indication that this condition will continue and that the MONTHLY may, within another year, become self-sustaining, even at the very low subscription price at which it is offered and with the very high degree of mechanical excellence at which it is maintained.

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MANAGING EDITOR.

Errata. In addition to the corrections noted on pages 104, 210, 258, the word *k-triple* in line 7 of Professor Dickson's article on page 84 should read *k-tuple*. Also in the second note on page 292, the name should be T. N. HAUN.

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